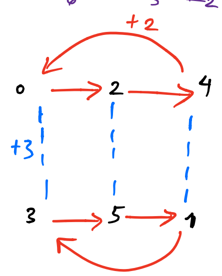
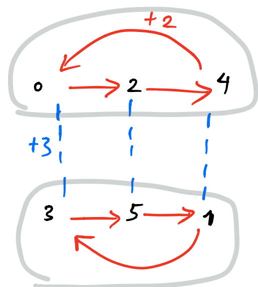
Sec 3.5 Quotient groups

Idea:

$\mathbb{Z}_6 \cong \mathbb{Z}_3 \times \mathbb{Z}_2$

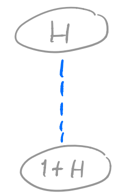




subgroup
$H = \{0,2,4\}$

$1+H = \{3,5,1\}$
a left coset of H

Collapse each coset into a single node, and get



a Cayley diagram for $C_2 \cong \mathbb{Z}_2$

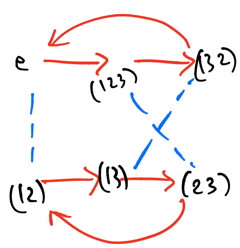
We denote this as

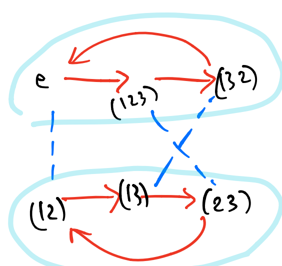
$\mathbb{Z}_6 / H \cong C_2$
"mod"

Connection to direct product:

$C_2 \times C_3 \cong C_6$
and
$H \cong C_3$
So
$C_6 / C_3 \cong C_2$

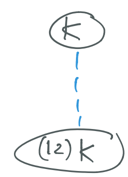
$S_3 \cong D_3$





subgroup
$K = \langle (123) \rangle$

$(12)K$
a left coset of K



Write $S_3 / K \cong C_2$
mod

Connection to direct product:

No connection.
$S_3 \not\cong C_2 \times C_3$
although
$K \cong C_3$.

<u>Obs</u>
- K is a normal subgroup of S_3
- H —— " —— of $\mathbb{Z}_6$
- It's possible to have $A/H \cong B/K$ where $A \not\cong B$ and $H \cong K$ (as in above example).

<u>Def</u> If $H \leq G$, denote $G/H \overset{\text{def}}{:=} \{\text{left cosets of } H \text{ in } G\}$
"just a set, not a group"
$= \{gH \mid g \in G\}$

Q: When is G/H a group? "Why did we care about normal subgroups?"

<u>Thm</u> If $H < G$,
G/H is a group if $H \trianglelefteq G$

Step 1: Define a binary operation $\cdot$ on G/H

If $aH, bH \in G/H$, define

$$aH \cdot bH := abH$$

Step 2: Verify that this def is well-defined, meaning it doesn't depend on our choice of coset representative.

Need to show: if $H \trianglelefteq G$,

if $a_1H = a_2H$ and $b_1H = b_2H$, then

$$a_1H \cdot b_1H = a_2H \cdot b_2H$$

Pf Suppose $H \trianglelefteq G$, $a_1H = a_2H$, $b_1H = b_2H$.

Then $a_1H \cdot b_1H = a_1b_1H$ by def

Think to self: want to see a_2b_2H at the end

$= a_1b_2H$ since $b_1H = b_2H$ by assumption

$= a_1Hb_2$ since $H \trianglelefteq G$ by assumption

$= a_2Hb_2$ since $a_1H = a_2H$ by assumption

$= a_2b_2H$ since $H \trianglelefteq G$ by assumption

$= a_2H \cdot b_2H$ by def $\square$

Step 3: Verify the 3 properties of a grp.

Identity: H is the identity elt in G/H

since $\left.\begin{array}{l} aH \cdot H = aeH = aH \\ H \cdot aH = eaH = aH \end{array}\right\} \forall aH \in G/H.$

Inverses: $(aH)^{-1}$ is $a^{-1}H$ because

$$aH \cdot a^{-1}H = aa^{-1}H = eH = H$$

Closure: $aH \cdot bH = abH \in G/H \quad \forall aH, bH \in G/H.$

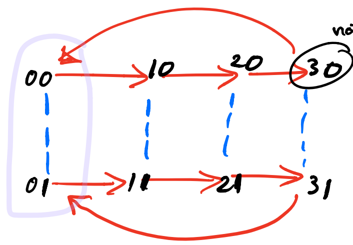
Def In this case, G/H is called a quotient group or factor group: G/H is the quotient group of G by the normal subgroup H.

Q: If $H \trianglelefteq G$, $K \trianglelefteq G$, and $H \cong K$,

are $G/H \cong G/K$?

Ex $G = \mathbb{Z}_4 \times \mathbb{Z}_2$

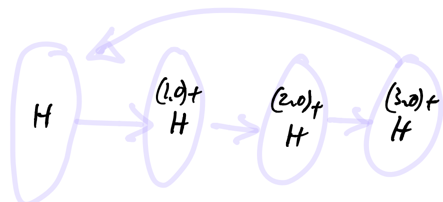
Note: G is abelian, so every subgroup is normal & you can always take quotient



notation $= (3,0) \in \mathbb{Z}_4 \times \mathbb{Z}_2$

"The quotient of $\mathbb{Z}_4 \times \mathbb{Z}_2$ by $H = \langle (0,1) \rangle$"

is $\mathbb{Z}_4 \times \mathbb{Z}_2 / \langle (0,1) \rangle$ has the Cayley diagram



Start here on Friday week 8

$\mathbb{Z}_4 \times \mathbb{Z}_2 / \langle (2,0) \rangle$ $K = \langle (2,0) \rangle = \{ (2,0), (0,0) \}$

Other 3 cosets:

$\underbrace{(1,0) + K = \{ (3,0), (1,0) \}}_{\text{has order 2 since}}$ $\underbrace{(0,1) + K = \{ (2,1), (0,1) \} \quad (1,1) + K = \{ (3,1), (1,1) \}}_{\text{also of order 2}}$

$(1,0) + K + (1,0) + K = (2,0) + K = \{ (0,0), (2,0) \} = K$

Has the same structure as $V_4 \not\cong C_4$

Answer to Q is NO.

Infinite example

$G = \mathbb{Z}$, $\underbrace{H = 2\mathbb{Z} = \{ \dots, -2, 0, 2, 4, \dots \}, \; K = 3\mathbb{Z}}_{\text{Normal subgroups}} = \{ 3z \mid z \in \mathbb{Z} \} = \{ \dots, 0, 3, 6, \dots \}$

Both H and K are infinite cyclic group

$\dots \to -2 \to 0 \to 2 \to 4 \to 6$ $\quad H \cong \langle a \mid \rangle$

$\dots \to -3 \to 0 \to 3 \to 6 \to \dots$ $\quad K \cong \langle a \mid \rangle$

But $G/2\mathbb{Z} = \{ 2\mathbb{Z}, 1 + 2\mathbb{Z} \} \cong C_2$

$G/3\mathbb{Z} = \{ 3\mathbb{Z}, 1 + 3\mathbb{Z}, 2 + 3\mathbb{Z} \} \cong C_3$