Sec 2.3 part II : Alternating Groups "Follow slides"

Rem There are many ways to write a permutation as a product of transpositions,

	# transpositions
E.g. $(45278) = (45)(42)(47)(48)$	4
$= (27)(28)(24)(25)$	4
$= (16)(78)(74)(75)(34)(72)(34)(16)$	8

Lemma 2 If the identity is written as the product of r transpositions, $Id = \tau_1 \tau_2 \cdots \tau_r$, then r is an even number.

(Proof was skipped during lecture)

Pf Since a transposition is not the identity, $r \neq 1$.

If $r = 2$, we are done.

So, suppose $r > 2$, and we proceed by induction.

Suppose the right most transposition is (ab).

Then, since $(ij) = (ji)$ $\forall\ i \neq j$,

the product $\tau_{r-1}\ \tau_r$ must be one of the following cases:

$$(ba)(ab) = Id$$

$$(ac)(ab) = (acb) = (ab)(bc)$$

$$(bc)(ab) = (abc) = (ac)(bc)$$

$$(cd)(ab) = (ab)(cd)$$

where a, b, c, d are distinct

"Note that "a" doesn't show up on the right-most 2-cycle"

If the first case occurs, delete $\tau_{r-1}\tau_r$ from the product to obtain $Id = \tau_1 \tau_2 \cdots \tau_{r-3}\, \tau_{r-2}$.
By induction, $r-2$ is even. Hence, r is even.

If one of the other 3 cases occurs, we replace $\tau_{r-1}\tau_r$ with the right-hand side of the corresponding equation to obtain a new product of r transpositions for Id.
In this new product, the occurrence of the integer "a" is in the 2nd-from-the-rightmost transposition τ_{r-1}

instead of the rightmost transposition τ_r.

Now repeat the procedure just described for $\tau_{r-1}\tau_r$ but this time for $\tau_{r-2}\,\tau_{r-1}$.

As before, we obtain a product of $(r-2)$ transpositions equal to Id (Case I)

or a new product of r transpositions (other three cases)

where the rightmost occurrence of "a" is in τ_{r-2}.

If the identity is the product of $r-2$ transpositions, then again we are done by our induction hypothesis.

Otherwise, we will repeat the procedure with $\tau_{r-3}\,\tau_{r-2}$.

At some point, either we will have two adjacent, identical transpositions canceling each other out or "a" will be shuffled so that it will appear only in the first transposition. However, the last case cannot occur, because the identity would not fix "a" in this situation.

Therefore, Id must be the product of $r-2$ transpositions and, again by our induction hypothesis, we are done.

Thm 3 If a permutation π can be expressed as the product of an odd # of transpositions, then any other product of transpositions equaling π must also contain an ~~even~~ # of transpositions.

Pf Suppose $\pi = \pi_1\,\pi_2 \cdots \pi_m \stackrel{*}{=} \tau_1\,\tau_2 \cdots \tau_r$, where m is odd. We must show that r is also odd.

Recall that the inverse of π is $\pi_m \pi_{m-1} \cdots \pi_1$.

Hence $Id = \underbrace{\pi}\, \underbrace{\pi_m \pi_{m-1} \cdots \pi_1}$
$\overset{*}{=} \tau_1 \tau_2 \cdots \tau_r \quad \pi_m \cdots \pi_1$

By Lemma 2, $r+m$ must be even.
Since m is odd, we can conclude that r is odd. $\boxed{\text{Thm 3}}$

Exercise Show: If π can be expressed as an even # of transpositions, then any other product of transpositions equaling π must also contain an odd # of transpositions.

[i.e. A permutation can be written w/ an even # of transpositions,
OR an odd # of transpositions,
but not both !]

Def
A perm is even if it can be expressed as an even # of transpositions
— " — odd ———— " ———— odd — " —

E.g. S_3: $(12)(23)$, $(23)(12)$ even; (13) odd, (12) odd, (23) odd, e

Exercise
Let $S = \{$odd permutations in $S_3\}$. Is it a group?
— " — $A_3 = \{$even — " — in $S_3\}$. — " —

— ended here Week 5 Friday —

Start here week 6 Monday

Def If $n \geq 2$, denote $A_n \overset{\text{def}}{:=} \{\text{even perms of } S_n\}$

Thm 4 A_n is a group (w/ composition as binary operation)

Pf • The product of two even permutations is an even perm

• The identity is an even perm since $e = (1\,2)(1\,2)$

• Exercise: Show that the inverse of an even perm is also even

A_3 has order 3. You've seen in HW 2 that there is only one possible multp table for a group of order 3, the cyclic group

$$\begin{tikzcd}
e \arrow[r, bend left] & (12)(23)=(132) \arrow[dl, bend left] \\
(132)^2=(123) \arrow[u] &
\end{tikzcd}$$

Q: How many even perms do you think are there in S_n?

Prop 5 Let $n \geq 2$. Let $B_n = \{\text{odd perms in } S_n\}$. Then $|A_n| = |B_n|$.

Pf Let $\tau := (1\,2) \in S_n$.

Define $f: A_n \longrightarrow B_n$

by $f(\pi) = \pi\,(1\,2)$ for all $\pi \in A_n$

First, show that f is surjective.

Let $b \in B_n$.

Let $x := b\,(12)$. Since b is an odd permutation, x is an even permutation.

Then $f(x) = x\,(12)$

$= b\,(12)(12)$

$= b$, as needed.

Prove that f is injective on your own.

To prove that f is injective,

suppose $f(x) = f(y)$ for some $x, y \in A_n$.

Then $x\,(12) = y\,(12)$.

So $x = x\,(12)(12)$

$= y\,(12)(12)$

$= y$. $\square$

Corollary $|A_n| = \dfrac{n!}{2}$

- Slide 1
  Briefly discuss the groups of symmetries of the 5 Platonic solids

Slide 1

Groups of symmetries of the Platonic solids

Shape	group
Tetrahedron	A_4
Cube Octahedron	S_4
Icosahedron Dodecahedron	A_5

- Slide 2

  Briefly explain that the Cayley diagram for the icosahedron, A_5, can be arranged on a truncated icosahedron if you choose a suitable set of minimal generating set.

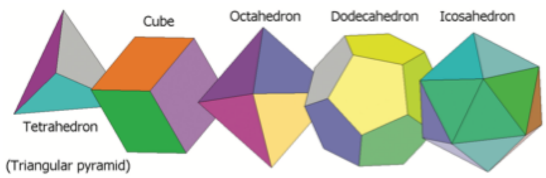
- The Cayley diagram for the dodecahedron, also A_5, can be arranged on a truncated dodecahedron if you choose a suitable set of minimal generating set.

Copy of Handout

Platonic solids

The symmetric groups and alternating groups arise throughout group theory. In particular, the groups of symmetries of the 5 Platonic solids are symmetric and alternating groups.

A 3-dimensional *Platonic solid* is a polytope with regular polygons as faces where all angles are equal and all sides are equal. There are only *five* 3-dimensional platonic solides:



The groups of symmetries of the Platonic solids are as follows:

shape	group
Tetrahedron	A_4
Cube	S_4
Octahedron	S_4
Icosahedron	A_5
Dodecahedron	A_5

Platonic solids

The Cayley diagrams for these 3 groups can be arranged in some very interesting configurations.

In particular, the Cayley diagram for Platonic solid 'X' can be arranged on a truncated 'X', where truncated refers to cutting off some corners.

For example, here are two representations for Cayley diagrams of A_5. At left is a truncated icosahedron and at right is a truncated dodecahedron.