Sec 2.3 Part I: Symmetric Groups

Def A <u>permutation</u> is an action that arranges a set of objects.
Usually, think of a permutation π as a bijection
$\pi : \{1, 2, \cdots, n\} \longrightarrow \{1, 2, \cdots, n\}$.

Two-line notation: $\pi = \begin{bmatrix} 1 & 2 & \cdots & n \\ \pi(1) & \pi(2) & \cdots & \pi(n) \end{bmatrix}$ e.g. $\pi = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{bmatrix}$ or $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{bmatrix}$ or $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{bmatrix}$

One-line notation: $\pi = [\pi(1)\pi(2)\cdots\pi(n)]$ e.g. $[2\,3\,4\,1]$ $[2\,1\,4\,3]$ $[3\,4\,1\,2]$ $[1\,3\,2\,4]$

Prop There are $\underline{n! = n\,(n-1)\,(n-2)\cdots 3 \cdot 2 \cdot 1}$ permutations of n objects
E.g. — " — $4! = 24$ permutations on 4 objects

Pf On the 1st pos, there are n choices,
— " — 2nd pos, — " — $n-1$ — " —, and so on.

picture: "depending on what we're doing, some notations are more convenient"

$\alpha = 1\ 2\ 3\ 4$ $\beta = 1\ 2\ 3\ 4$ $1\ 2\ 3\ 4$ $1\ 2\ 3\ 4$

Let $S_n = \{$all permutations on n objects$\}$
To make S_n a group, we need

1. (Associative binary operation)

Combining permutations:

$\alpha\ \beta$ means do α then β: $1\ 2\ 3\ 4$
Read left to right

$\beta\ \alpha$ means do β then α: $1\ 2\ 3\ 4$

2. (Identity Elt) picture: $1\ 2\ 3\ 4$ two-line notation $\begin{bmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 1 & 2 & 3 & & \cdots & n \end{bmatrix}$

$id : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$
where $x \longmapsto x$

3. (Every elt should have an inverse)
Since a permutation is a bijection, it has an inverse.
E.g. The inverse of $1\ 2\ 3\ 4$ is $1\ 2\ 3\ 4$

— " — $1\ 2\ 3\ 4$ is $1\ 2\ 3\ 4$

$\therefore$ S_n is a group, called the $\underset{\smile}{\text{Symmetric group}}$.

<u>Cycle notation</u>

$\alpha = 1\ 2\ 3\ 4$ $\beta = 1\ 2\ 3\ 4$ $1\ 2\ 3\ 4$ $1\ 2\ 3\ 4$

$(1\,2\,3\,4)$ $(1\,2)(3\,4)$ $(1\,3)(2\,4)$ $(1)(2\,3)(4) = (2\,3)$
$= (2\,3\,4\,1)$
$= (3\,4\,1\,2)$

<u>Practice</u>

$1\ 2\ 3$ $1\ 2\ 3$

- $(1\,2)(2\,3) = (1\,3\,2)$
- $(1\,2\,3\,4)\ (1\,3)(2\,4) = (1\,4\,3\,2)$

$\Big\}$ $(1\,3\,2)\,(1\,2) = (1\,3)\,(2)$

Claim $\overset{\curvearrowright}{1\,2}\,3$ & $1\,\overset{\curvearrowright}{2\,3}$ generate S_3 start here WED

(12) (23)

Pf S_3 has $3! = 6$ elements

$Id = (12)(12)$

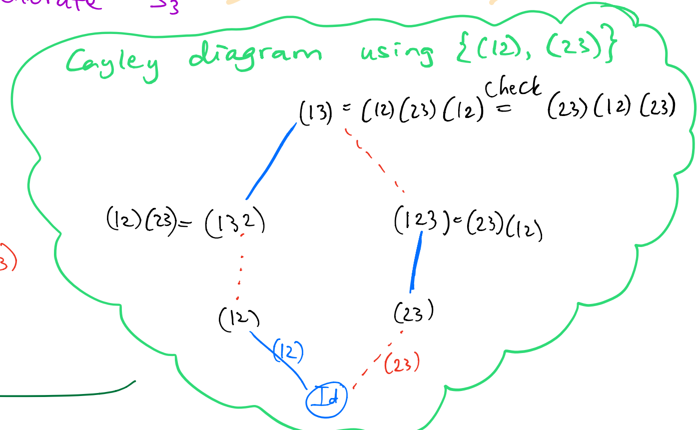
(12) ✓

(23) ✓

$(13) \stackrel{?}{=} (12)(23)(12)$ ← Note $(132)(12) = (13)$

$(123) = (23)(12)$

$(132) = (12)(23)$ from before



Hence, S_3 is the same group as D_3 :)

Exercise: Find two generators a, b so $a^3 = Id$, $b^2 = Id$, $aba = b$ } Hint: look @ Cayley graph, take (132) or (123)

we know this is possible because $D_3 = \langle a, b \mid a^3 = Id,\ b^2 = Id,\ aba = b \rangle$

Observations

① Every permutation can be decomposed into a product of disjoint cycles. E.g. $\pi = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & & & & & & & & \end{bmatrix}$ ← Tell me your fave #

② Disjoint cycles commute

$(1\,4\,6\,5)\,(2\,3)\,(8\,10\,9) = (23)\,(8, 10, 9)(1\,4\,6\,5)$.

Def A permutation that swaps two objects & fixes the rest, i.e. $(i\,j)$ in cycle notation and

$1,\ 2\ \cdots\ i,\ i+1,\ \cdots\ j-1,\ j\ \cdots,\ n-1,\ n$

is called a <u>transposition</u>.

Thm 1 The group S_n is generated by transpositions (2-cycles)

First, intuition of Pf:

If I have 10 of you sit in a row, & I want to rearrange you in some other way, I can achieve this by successively having a pair of people swap seats.

e.g. $(4\,5\,2\,7\,8) = (45)(42)(47)(48)$ this is not unique —
Check

Exercise Find other ways to do this

ended here WK 5 wed

Pf Since every perm can be written as a product of disjoint cycle, it's enough to show that we can write any cycle as a product of transpositions.

Let $(a_1\,a_2 \ldots a_k)$ be a cycle in S_n.

Then $(a_1, a_2)(a_1, a_3) \cdots (a_1, a_k) = (a_1 a_2 a_3 a_4 \cdots a_k)$. $\square$

Remark In fact, the adjacent transpositions $\{(1,2),(2,3),\ldots,(n-1,n)\}$ form a minimal gen set for S_n.

Ex "Look @ adjacent pairs which are out of order"

$[1\ 3\ 5\ 2\ 4] = (1)(2\,3\,5\,4)$

$\downarrow (34)$ multiply on left by (34)

$[1\ 3\ 2\ 5\ 4]$

$\downarrow (45)$

$[1\ 3\ 2\ 4\ 5]$

$\downarrow (23)$

$[1\ 2\ 3\ 4\ 5] = Id$

Means $[1\,3\,5\,2\,4] = (34)(45)(23)$

left to right