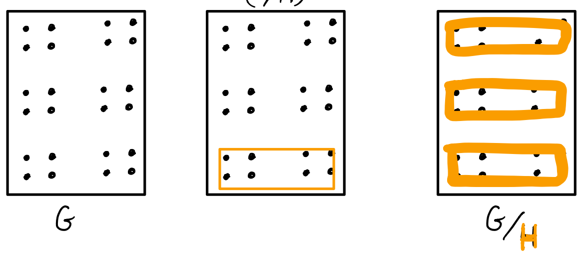
Sec 3 The 3rd Isomorphism Thm

- Statement:
  Let N and H be normal subgroups of G, with $N < H$.
  Then (i) The quotient group H/N is normal in G/N

  (ii) $(G/N)/(H/N) \cong G/H$

- 

- Define $\varphi : G/N \longrightarrow G/H$ by
  $$gN \longmapsto gH$$

2 Prove that φ is well-defined:
We need to show that if $aN = bN$ then $\varphi(aN) = \varphi(bN)$.

Suppose $aN = bN$. Then $a \overset{(*)}{=} bn$ for some $n \in N$.

Then $\varphi(aN) = aH$ by def of φ
$= bnH$ by $(*)$
$= bH$ since $n \in N \subset H$
$= \varphi(bN)$ by def of φ $\boxed{2}$

4 Prove that φ is a homomorphism:

We need to show that $\varphi(aN \cdot bN) = \varphi(aN)\,\varphi(bN)$ for all $aN, bN \in G/N$.

$\varphi(aN \cdot bN) = \varphi(abN)$ by def of binary operation in quotient group G/N

$= abH$ by def of φ

$= aH \cdot bH$ by def of binary operation in quotient group G/H.

$= \varphi(aN)\,\varphi(bN)$

$\boxed{4}$

1 and 5

Prove (i) $H/N \trianglelefteq G/N$:

We will first show that $H/N = \ker(\varphi)$ and apply the theorem which says that the kernel of every homomorphism f is a normal subgroup of the domain of f.

Recall that $H/N \overset{\text{def}}{:=} \{\text{left cosets of } N \text{ in } H\} = \{xN \mid x \in H\}$.

To show that $H/N \subset \ker(\varphi)$, note that

if $h \in H$, then $\varphi(hN) = hH$
$= H,$

which is the identity element in the quotient group $\overbrace{G/H}^{\text{codomain of } \varphi}$

To show that $\ker(\varphi) \subset H/N$,

suppose $x \in G$ where $\varphi(xN) = H$.

But $\varphi(xN) = xH$ by def of φ, so $xH = H$.

So $x \in H$.

So $xH \in H/N$.

Hence $\ker(\varphi) = H/N$.

Since $\ker(\varphi) \trianglelefteq \underbrace{G/N}_{\text{domain of homomorphism } \varphi}$, $H/N \trianglelefteq G/N$.

$\boxed{1}$

3. Prove that $\varphi : G/N \longrightarrow G/H$ is surjective:
$gN \longmapsto gH$

Suppose $xH \in G/H$ is a left coset of H in G.

Then $xN \in G/N$ is a left coset of N in G,

where $\varphi(xN) = xH$ by def of φ $\boxed{3}$

6. By the 1st Iso Thm, $\dfrac{G/N}{\ker\varphi} \cong \operatorname{Im}(\varphi)$

Since φ is surjective, $\operatorname{Im}(\varphi) = G/H$.

Since $\ker\varphi = H/N$, we have $(G/N)/(H/N) \cong G/H$