

Sec 2: 2nd Isomorphism Thm (Diamond Isomorphism Thm)

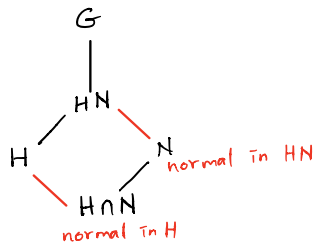
• Statement: Let $H < G$, $N \triangleleft G$. Then

(i) $HN \stackrel{\text{def}}{=} \{hn \mid h \in H, n \in N\}$ is a subgroup of G

(ii) $H \cap N$ is a normal subgroup of H

(iii) The quotient group $\frac{HN}{N}$ is isomorphic to the quotient group $H/(H \cap N)$

• "diamond" lattice



• Strategy of proof: Define a (surjective) map

$$\phi: \overset{\text{domain}}{H} \longrightarrow \overset{\text{codomain}}{HN/N} \quad \text{by}$$

$$\phi: h \mapsto hN$$

and apply 1st Isomorphism Thm

1 Prove that (i) HN is a subgroup of G :

We need to show that $e \in HN$

- For all $h_1 n_1, h_2 n_2 \in HN$, we have $h_1 n_1 h_2 n_2 \in HN$.
- For all $hn \in HN$, $(hn)^{-1} \in HN$.

• Since H and N are subgroups of G , they both contain the identity elt, so $e \in HN$

• Suppose $h_1 n_1, h_2 n_2 \in HN$. Then

$$(h_1 n_1)(h_2 n_2) = h_1 \underbrace{h_2 h_2^{-1}}_{\text{new}} n_1 h_2 n_2$$

$$= h_1 h_2 (h_2^{-1} n_1 h_2) n_2$$

$$\in HN \quad \text{because } h_2^{-1} n_1 h_2 \in N \text{ (since } N \triangleleft G \text{)}.$$

• Suppose $hn \in HN$. Then the inverse of hn is $n^{-1} h^{-1}$.

To show that $n^{-1} h^{-1} \in HN$, note that

$$n^{-1} h^{-1} = \underbrace{h^{-1} h}_{\text{new}} n^{-1} h^{-1}$$

$$= h^{-1} (h n^{-1} h^{-1})$$

$$\in HN \quad \text{because } h n^{-1} h^{-1} \in N \text{ (since } N \triangleleft G \text{)}. \quad \square$$

2 Prove that (ii) the intersection $H \cap N \triangleleft H$:

We need to show that $h^{-1}nh \in H \cap N$ for all $h \in H, n \in H \cap N$.

Let $h \in H$ and $n \in H \cap N$.

Then $h^{-1}nh \in H$ since $h^{-1}, n, h \in H$ and H is a group.
Also, $h^{-1}nh \in N$ since N is normal in G .

Hence $h^{-1}nh \in H \cap N$ 2

3 Prove that $\phi: H \rightarrow HN/N$ is a homomorphism:

$$\phi(h) = hN$$

We need to show that $\phi(ab) = \phi(a)\phi(b)$.

Recall from def of quotient groups that $aN \cdot bN = abN$.

$$\phi(ab) = abN$$

$$= aN \cdot bN \text{ by def of binary operation of a quotient group}$$

$$= \phi(a)\phi(b)$$

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4 Prove that $\phi: H \rightarrow HN/N, \phi(h) = hN$ is surjective:

We need to show that, for any $xN \in HN/N$, there is $h \in H$ with $\phi(h) = xN$.

Note that any coset in the codomain HN/N can be written

as hnN where $h \in H, n \in N$

but $hnN = hN$, so $\phi(h) = hN = hnN$.

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5 Prove that $\ker(\phi) = H \cap N$:

We need to show that $\phi(n) = N$ (since N is the identity in the quotient group HN/N)
for all $n \in H \cap N$

and $\phi(x) \neq N$ if $x \notin H \cap N$.

$$\phi(n) = nN = N \text{ for all } n \in H \cap N$$

Suppose $x \in H$ and $x \notin N$.

domain of ϕ

Then $\phi(x) = xN \neq N$ (since we know that two distinct left cosets are disjoint)

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6 By the 1st Isomorphism Thm,

$$H/\ker(\phi) \cong \text{Im}(\phi)$$

Since $\ker(\phi) = H \cap N$ and $\text{Im}(\phi) = HN/N$,

we have
$$H/\ker(\phi) \cong HN/N$$

□