# Sec 2: 2nd Isomorphism Thm (Diamond Isomorphism Thm)

- Statement: Let $H < G$, $N \trianglelefteq G$. Then
  (i) $HN \overset{\text{def}}{:=} \{hn \mid h \in H, n \in N\}$ is a subgroup of $G$
  (ii) $H \cap N$ is a normal subgroup of $H$
  (iii) The quotient group $HN/N$ is isomorphic to the quotient group $H/(H\cap N)$

- "diamond" lattice

$$\begin{tikzcd}
 & G \arrow[d, dash] & \\
 & HN \arrow[dl, dash] \arrow[dr, dash] & \\
H \arrow[dr, dash, "\text{normal in } H"'] & & N \arrow[dl, dash] \text{ normal in } HN \\
 & H\cap N &
\end{tikzcd}$$

- Strategy of proof: Define a (surjective) map

$$\phi : \underset{\text{domain}}{H} \longrightarrow \underset{\text{codomain}}{HN/N} \quad \text{by}$$

$$\phi : h \longmapsto hN$$

and apply 1st Isomorphism Thm

1 Prove that (i) $HN$ is a subgroup of $G$:

We need to show that
- $e \in HN$
- For all $h_1n_1, h_2n_2 \in HN$, we have $h_1n_1h_2n_2 \in HN$.
- For all $hn \in HN$, $(hn)^{-1} \in HN$.

- Since $H$ and $N$ are subgroups of $G$, they both contain the identity elt, so $e \in HN$
- Suppose $h_1n_1, h_2n_2 \in HN$. Then

$$(h_1n_1)(h_2n_2) = h_1 \underbrace{h_2h_2^{-1}}_{\text{new}} n_1 h_2 n_2$$
$$= h_1h_2(h_2^{-1}n_1h_2)n_2$$
$$\in HN \qquad \text{because } h_2^{-1}n_1h_2 \in N \text{ (since } N \trianglelefteq G\text{).}$$

- Suppose $hn \in HN$. Then the inverse of $hn$ is $n^{-1}h^{-1}$.
  To show that $n^{-1}h^{-1} \in HN$, note that

$$n^{-1}h^{-1} = \underbrace{h^{-1}h}_{\text{new}} n^{-1}h^{-1}$$
$$= h^{-1}(hn^{-1}h^{-1})$$
$$\in HN \qquad \text{because } hn^{-1}h^{-1} \in N \text{ (since } N \trianglelefteq G\text{).} \quad \boxed{1}$$

2 Prove that (ii) the intersection $H \cap N \trianglelefteq H$:

We need to show that $h^{-1} n h \in H \cap N$ for all $h \in H$, $n \in H \cap N$.

Let $h \in H$ and $n \in H \cap N$.

Then $h^{-1} n h \in H$ since $h^{-1}, n, h \in H$ and $H$ is a group.

Also, $h^{-1} n h \in N$ since $N$ is normal in $G$.

Hence $h^{-1} n h \in H \cap N$ $\boxed{2}$

3 Prove that $\phi : H \longrightarrow HN/N$ is a homomorphism:

$$\phi(h) = hN$$

We need to show that $\phi(ab) = \phi(a)\phi(b)$.

Recall from def of quotient groups that $aN \cdot bN = abN$.

$$\phi(ab) = abN$$

$= aN \cdot bN$ by def of binary operation of a quotient group

$= \phi(a)\phi(b)$

$\boxed{3}$

4 Prove that $\phi : H \to HN/N$, $\phi(h) = hN$ is surjective:

We need to show that, for any $xN \in HN/N$, there is $h \in H$ with $\phi(h) = xN$.

Note that any coset in the codomain $HN/N$ can be written

as $hnN$ where $h \in H$, $n \in N$

but $hnN = hN$, so $\phi(h) = hN = hnN$. $\boxed{4}$

5 Prove that $\ker(\phi) = H \cap N$:

We need to show that $\phi(n) = N$ for all $n \in H \cap N$ (since $N$ is the identity in the quotient group $HN/N$)

and $\phi(x) \neq N$ if $x \notin H \cap N$.

$\phi(n) = nN = N$ for all $n \in H \cap N$

Suppose $x \in \underbrace{H}_{\text{domain of } \phi}$ and $x \notin N$.

Then $\phi(x) = xN \neq N$ (since we know that two distinct left cosets are disjoint)

$\boxed{5}$

6 By the 1st Isomorphism Thm,

$$H/\ker(\phi) \cong \mathrm{Im}(\phi)$$

Since $\ker(\phi) = H \cap N$ and $\mathrm{Im}(\phi) = HN/N$,

we have $$H/H \cap N \cong HN/N$$

$\boxed{6}$