Name: ____ **Abstract Algebra**
Quiz 1 Sample

1. A group's generators have a special status in a Cayley diagram for the group. What is that special status?

> **Solution:** The generators are represented by the diagram's arrows.

2. What do the arrows in a Cayley diagram represent?

> **Solution:** Each type of arrow (distinguished by color or label) represents a generator.

3. What do the nodes in a Cayley diagram represent?

> **Solution:** Each node represents an action in the group (so the number of nodes of the Cayley diagram is equal to the number of actions in the group).

4. (1) Give two possible generators (in a minimal generating set) for the rectangle puzzle. (2) What other actions are there besides the generators? Used verbs to describe each action.

> **Solution:** (1) A possible answer: The two generators can be the horizontal flip and vertical flip.
> (Other possible answers: The two generators can be the horizontal flip and the rotation by 180^o; the two generators can be the vertical flip and the rotation by 180^o)
> (2) The other actions are the "do-nothing" action and the combined action "horizontal flip then vertical flip" which corresponds to a rotation by 180^o.

5. Provide 2 examples of a group. In each case, describe a set of generators. For one of your examples, draw a Cayley diagram.

6. Let r be a clockwise rotation by $2\pi/6$ radians (60^o) of a regular 6-gon. This generates a group denoted by $C_6 = \langle r \rangle$ which consists of the 6 rotating actions $\{e, r, r^2, r^3, r^4, r^5\}$.

   a.) Draw the original configuration of the hexagon and also the other 5 configurations that you would get after applying the 5 non-identity rotations.
   b.) Draw a Cayley diagram for C_6 with $\{r\}$ as the generating set.
   c.) Is the group $\langle r^5 \rangle$ (generated by a 300^o rotation) the same as C_6?
   d.) List the actions in the group $\langle r^2 \rangle$ (generated by a 120^o rotation).
   e.) List the group $\langle r^3 \rangle$ (generated by a 180^o rotation).
   f.) Is $\{r^2\}$ a generating set of C_6? What about $\{r^3\}$?
   g.) How many elements does the group $\langle r^3, r^4 \rangle$ have? What about $\langle r^3, r^2 \rangle$?
   h.) Give a possible group presentation for this group.

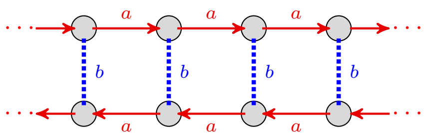
   > **Solution:** $\langle a | a^6 = e \rangle$, $\langle a, b | a^3 = e, b^2 = e, ab = ba \rangle$ (To see why the second one is a group presentation, try drawing a Cayley diagram with two generators).

7. What does it mean for a group to be abelian?

Solution: Answer: It means that $ab = ba$ for every action a, b in the group.

Side Note: It is enough to check that $ab = ba$ for every generator a, b in the group. In particular, this means that if a group G can be generated by just one action then G must be abelian.

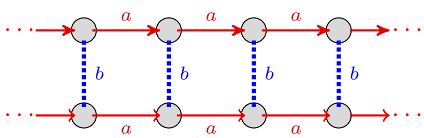
8. Write a possible group presentation using the following Cayley diagram.
   Is the group with this Cayley diagram abelian? Explain.



Solution: $\langle a, b \mid b^2 = 1, abab = e\rangle$ or $\langle a, b \mid b^2 = 1, ab = ba^{-1}\rangle$ or $\langle a, b \mid b^{-1} = b, ab = ba^{-1}\rangle$, among other possibilities

This group is no abelian; for example ab does not equal ba.

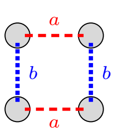
9. Write a possible group presentation using the following Cayley diagram.
   Is the group with this Cayley diagram abelian? Explain.



Solution: $\langle a, b \mid b^2 = 1, aba^{-1}b = e\rangle$ or $\langle a, b \mid b^2 = 1, ab = ba\rangle$, among other possibilities.

The group is abelian; the two generators commute with each other, $ab = ba$.

10. Describe a group of actions with the following as a Cayley diagram. Is it abelian? Write a possible group presentation.



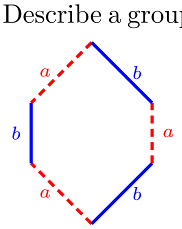
Solution: Hint: Describe the "rectangle puzzle" actions (see slides 1.1) or "light-switch" actions (see slides 1.1).

Hint: To show that a group is abelian, it is enough to show that $xy = yx$ for all generators x, y.

Hint: To show that a group is not abelian, it is enough to find two actions c, d such that $cd \neq dc$.

A possible group presentation is given in slides 1.3.

11. Describe a group of actions with the following as a Cayley diagram. Is it abelian? Write a possible group presentation.



Solution: Hint: See HW 1 and slides 1.3.

Hint: To show that a group is abelian, it is enough to show that $xy = yx$ for all generators x, y.

Hint: To show that a group is not abelian, it is enough to find two actions c, d such that $cd \neq dc$.

A possible group presentation for this group is given in slides 1.3.