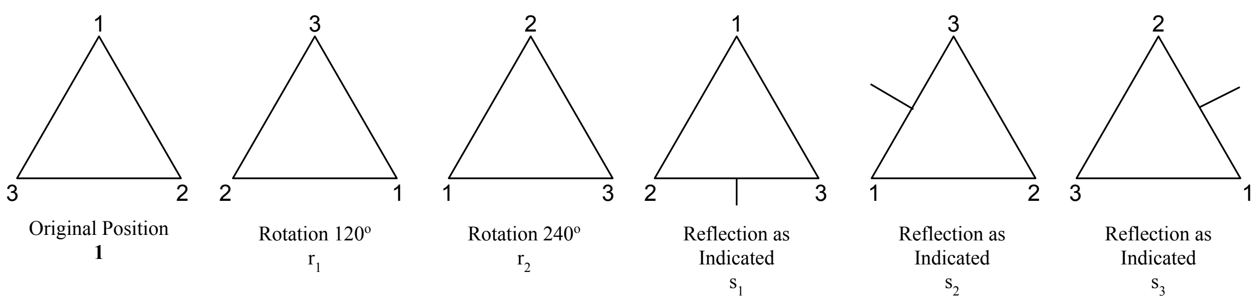
Symmetries

A **symmetry** of a geometric figure is a rearrangement of the figure preserving the arrangement of its sides and vertices as well as its distances and angles. A map from the plane to itself preserving the symmetry of an object is called a **rigid motion**. (We can extend this idea to symmetries of figures in three dimensions.)

It is the final position of the figure that is important, not the motion itself. For example, if we rotate the triangle below through 120° or 480° the triangle ends up in the same final position, so we do not think of these as distinct symmetries.

Consider the symmetries of an equilateral triangle, as illustrated below. Each sketch shows the resulting position when the specified motion is applied to the triangle starting in the original position.



We can apply one of the rigid motions, and then, *continuing from the new position of the triangle*, apply another of the rigid motions. We can then record the overall effect as one of six symmetries listed above.

For example, if we first apply s_1, then (continuing from the resulting position) apply r_1 (a 120° clockwise rotation) we end up with s_3. Using our usual function composition notation, we can think of this as $r_1 \circ s_1$. In fact, we write $r_1 \circ s_1 = s_3$.

It is useful to write the results of all such combinations in a **Cayley table**, as shown below. We show the result of $r_1 \circ s_1$ in the row labelled r_1 and the column labelled s_1.

$\circ$	1	r_1	r_2	s_1	s_2	s_3
1						
r_1				s_3		
r_2						
s_1						
s_2						
s_3						

You have been given an equilateral triangle with the vertices labeled 1, 2, 3 on both sides. Use the cut-out triangle to determine the compositions and use this to complete the table. We will denote this collection of symmetries, together with composition, by D_3.

1. Give an example from the table that shows that the order in which you apply the rigid motions matters. Describe this both using our notation and geometrically.

2. Are there examples from the table for which the order doesn't matter? If so, list some.

3. Does r_1 have an inverse, i.e. is there another motion we can compose it with to get back to the original position? If so, describe it both using our notation and geometrically.

4. Are there any elements without an inverse? If so, list some.

5. Let D_4 denote the symmetries of a square. How many symmetries are there? For the collection of symmetries of a non-square rectangle, see the article Group Think by Steven Strogatz

Clock Arithmetic

Consider the numbers on a clock, and imagine 0 in place of 12. We will denote this set by $\mathbb{Z}_{12}$. So $\mathbb{Z}_{12} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$.

We define "addition modulo 12" on this set as follows: if $a, b \in \mathbb{Z}_{12}$, then $a + b \pmod{12}$ is the hour on the clock-face that is b hours after a. For example, $9 + 5 \equiv 2 \pmod{12}$ (since 0 is 3 hours after 9, and we need an additional 2 hours after that.)

We can also define "multiplication mod 12" on $\mathbb{Z}_{12}$ by thinking of this as repeated addition modulo 12. So for $a, b \in \mathbb{Z}_{12}$, we think of $ab \pmod{12}$ as the result of adding b to itself a times, modulo 12. For example $(3)(7) \equiv 9 \pmod{12}$.

There is nothing special about 12 here; we can just as easily define addition and multiplication mod n on the set $\{0, 1, 2, \ldots, n-1\}$ for any fixed positive integer n. Simply imagine a clock-face with the numbers $\{0, 1, 2, \ldots, n-1\}$ in place of $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$, and for a and b in this set, define $a + b \pmod{n}$ to be the hour on this clock-face that is b hours after a . Define $ab \pmod{n}$ to be the result of adding b to itself a times, modulo n.

We can create tables for these operations, just as we did for symmetries. Let $\mathbb{Z}_4 = \{0, 1, 2, 3\}$. Draw up a table for addition mod 4, and a separate table for multiplication mod 4.

$+$	**0**	**1**	**2**	**3**
0				
1				
2				
3			1	

$\times$	**0**	**1**	**2**	**3**
0				
1				
2				
3			2	

1. Does each element have an additive inverse, i.e. is there another number we can add to it get 0?

2. Does each element have a multiplicative inverse, i.e. is there another number we can multiply it with to get 1?

3. Both addition and multiplication mod 4 are commutative. How can you see this in the tables?