IN-CLASS WORKSHEET FOR HOMEWORK 5

Theorem 1 (Theorem 3 from Slides 3.3)**.** Let H be a subgroup of G. Then the following are all equivalent.

(i) $gH = Hg$ for all $g \in G$; ("left cosets are right cosets");

(ii) $gHg^{-1} = H$ for all $g \in G$; ("only one conjugate subgroup")

(iii) $ghg^{-1} \in H$ for all $h \in H, g \in G$; ("closed under conjugation").

(iv) The subgroup H is normal in G.

a. The *center* of a group G is the set

$$Z(G) = \{z \in G \mid gz = zg,\ \forall g \in G\} = \{z \in G \mid gzg^{-1} = z,\ \forall g \in G\}.$$

- Prove that $Z(G)$ is normal in G using part (i), (ii), or (iii) of the above theorem. Attempt to choose the part that would minimize your work.
- Compute the center of the following groups: D_4, D_5, D_6, D_7 D_n.

  Hint: Draw the Cayley diagram using two reflections (with angle π/n between the two mirrors) as minimal generators for $n = 4, 5$. For example, for $n = 4$, use the vertical mirror and one of the diagonal mirrors (so the degree between the mirrors is 45^o).
- Compute the center of Q_8.

  Recall that the elements of the Quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ are governed by the rules $i^2 = j^2 = k^2 = -1, \quad ij = k, \quad jk = i, \quad ki = j, \quad ji = -k, \quad kj = -i, \quad ik = -j$.
- Compute the center of A_4

  Hint: A non-identity permutation in S_4 is an even permutation if and only of its cycle notation is of the form $(ab)(cd)$ or (abc). (Make sure you know why!)

  Do $(ab)(cd)$ and (abc) commute?
- Compute the center of S_4.

  Hint: Every non-identity permutation in S_4 can be written in the form (ab), (abc), $(abcd)$, and $(ab)(cd)$. Can you find a permutation that does not commute with (ab)? With $(abcd)$?
- Prove or disprove that "the center of a direct product is the direct product of the centers", that is, $Z(A \times B) = Z(A) \times Z(B)$.
- Use what you've done so far to compute the center of $D_n \times Q_8$. Draw the Cayley diagram for $Z(D_n \times Q_8)$

b. Find a chain of 3 distinct subgroups $K \leq H \leq G$ where $K \trianglelefteq H \trianglelefteq G$ but K is not a normal in G.

Hint: The non-abelian group of order 6 (S_3 or D_3) is too small to produce this example because the maximum chain of distinct subgroups $\{e\} \leq H \leq D_3$, and $\{e\}$ is always normal. There is no non-abelian group of order 7, so try to find an example within a non-abelian group of order 8.

c. Let H be a normal subgroup of G. Given two fixed elements $a, b \in G$, define the sets

$$aHbH := \{ah_1bh_2 \mid h_1, h_2 \in H\} \qquad \text{and} \qquad abH := \{abh \mid h \in H\}.$$

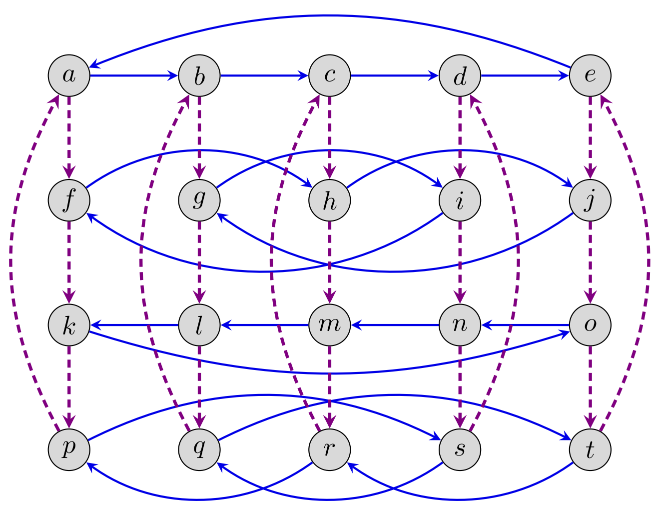
Prove that $aHbH \subset abH$.

d.

Definition 2. The set of elements in G that vote in favor of H's normality is called the *normalizer of* H *in* G, denoted $N_G(H)$. That is,

$$N_G(H) = \{g \in G : gH = Hg\} = \{g \in G : gHg^{-1} = H\}.$$

Let G be the group whose Cayley diagram is shown below, and suppose e is the identity element. Consider the subgroups $A = \langle a \rangle = \{a, b, c, d, e\}$ and $J = \langle j \rangle = \{e, j, o, t\}$.



Carry out the following steps for both of the subgroups A and J. List the cosets element-wise.

(a) Write G as a disjoint union of the subgroup's left cosets.

(b) Write G as a disjoint union of the subgroup's right cosets.

(c) Use your coset computation to immediately compute the normalizer of the subgroup. Based on the computation for the normalizer, what you can say about this subgroup?

(d) If G/A is a group, perform the quotient process and draw the resulting Cayley diagram for G/A. If G/J is a group, perform the quotient process and draw the resulting Cayley diagram for G/J.