Math3230 Abstract Algebra Homework 5

FIRST AND LAST NAME

Before you begin, review Section 3.2 (cosets), 3.3 (Normal subgroups), 3.4 (direct products), and 3.5 (quotients). Skim relevant parts of Chapter 6 and 7 of *Visual Group Theory* (VGT): `https://ebookcentral.proquest.com/lib/UCONN/detail.action?docID=3330331`.

Note that you have worked on a few of the exercises during class. There are a few hints in the .tex source code (commented out).

Write up solutions to all exercises.

1. Draw the subgroup lattice of the alternating group $A_4 = \langle (1\,2\,3), (1\,2)(3\,4) \rangle$. Then carry out the following steps for two of its subgroups, $H = \langle (1\,2\,3) \rangle$ and $K = \langle (1\,2)(3\,4) \rangle$. When writing a coset, list all of its elements, for example $(12)(34)H = \{\dots\}$.

   (a) Write A_4 as a disjoint union of the subgroup's left cosets.

   (b) Write A_4 as a disjoint union of the subgroup's right cosets.

   (c) Find all conjugates of the subgroup, and determine whether it is normal in A_4.

   Hint: Check the subgroups with Group Explorer. We did part of this question during class during Slides 3.3 class.

2. The *center* of a group G is the set

$$Z(G) = \{z \in G \mid gz = zg,\ \forall g \in G\} = \{z \in G \mid gzg^{-1} = z,\ \forall g \in G\}.$$

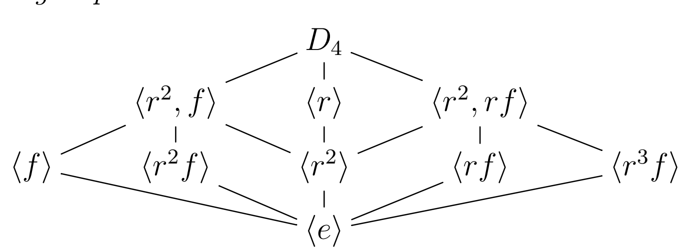
   (a) Prove that $Z(G)$ is a subgroup of G.

   (b) Prove that $Z(G)$ is normal in G.

   (c) Compute the center of the following groups: C_6, D_4, D_5, Q_8, A_4, S_4, and $D_4 \times Q_8$. See in-class worksheet for extra hints.

I: First, review the definition of a *conjugate subgroup* in Slides 3.3.

The subgroup lattice of D_4 is shown here:



For each of the 10 subgroups of D_4, find all of its conjugates, and determine whether it is normal in D_4. Fully justify your answers. [*Hint*: do this without computing xHx^{-1} for any subgroup H.]

II: Consider a chain of subgroups $K \leq H \leq G$.

   (a) Prove or disprove (with a counterexample): If $K \trianglelefteq G$, then $K \trianglelefteq H$.

   (b) Prove or disprove (with a counterexample): If $K \trianglelefteq H \trianglelefteq G$, then $K \trianglelefteq G$.

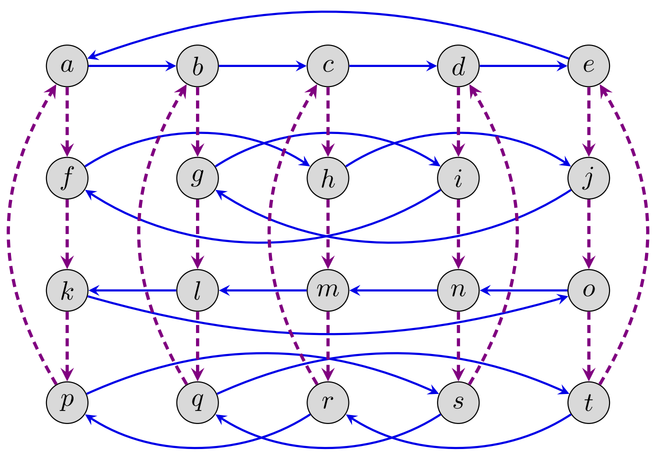
3. Let H be a subgroup of G. Given two fixed elements $a, b \in G$, define the sets

$$aHbH = \{ah_1bh_2 \mid h_1, h_2 \in H\} \qquad \text{and} \qquad abH = \{abh \mid h \in H\}.$$

Prove that if $H \trianglelefteq G$, then $aHbH = abH$.

4. Prove that $A \times \{e_B\}$ is a normal subgroup of $A \times B$, where e_B is the identity element of B. That is, show first that it is a subgroup, and then that it is normal.

5. Let G be the group whose Cayley diagram is shown below, and suppose e is the identity element. Consider the subgroups $A = \langle a \rangle = \{a, b, c, d, e\}$ and $J = \langle j \rangle = \{e, j, o, t\}$.



Carry out the following steps for both of the subgroups A and J. List the cosets element-wise.

(a) Write G as a disjoint union of the subgroup's left cosets.

(b) Write G as a disjoint union of the subgroup's right cosets.

(c) Compute the normalizer of the subgroup (see Sec 3.6 lecture notes or slides for definition). Write a short sentence about what you can say about this subgroup based on your computation.

(d) First, review the quotient process from Lecture notes Sec 3.5 "Quotient groups" (or Slides 3.5). If possible (that is, if this subgroup is normal in G), apply the quotient process, shrinking the left cosets into individual nodes. Draw the resulting diagram and state which familiar group the quotient is isomorphic to.

(e) Find all conjugates of the subgroup.

III: Let H be a subgroup of an abelian group G. Prove that H and G/H are both abelian.

6. First, review Section 3.5 "Quotient groups" of lecture notes or slides (or watch the linked video by M. Macauley).

All of the following statements are *false*. For each one, exhibit an explicit counterexample, and justify your reasoning. Assume that each $H_1 \trianglelefteq G_1$ and $H_2 \trianglelefteq G_2$.

(a) If H and G/H is abelian, then G is abelian.

(b) If every proper subgroup H of a group G is cyclic, then G is cyclic.

(c) If $G_1 \cong G_2$ and $H_1 \cong H_2$, then $G_1/H_1 \cong G_2/H_2$.

(d) If $G_1 \cong G_2$ and $G_1/H_1 \cong G_2/H_2$, then $H_1 \cong H_2$.

(e) If $H_1 \cong H_2$ and $G_1/H_1 \cong G_2/H_2$, then $G_1 \cong G_2$.