Math3230 Abstract Algebra Homework 4

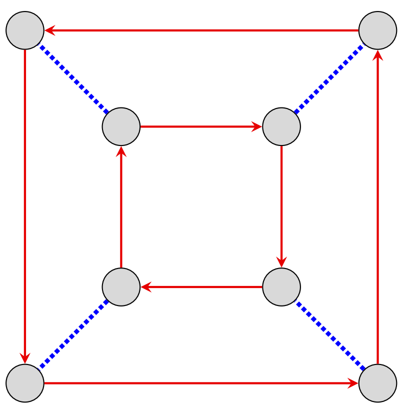
FIRST AND LAST NAME

Read the following from *Visual Group Theory* (VGT): https://ebookcentral.proquest.com/lib/UCONN/detail.action?docID=3330331: VGT Sec 6.1 (regularity of Cayley diagram) and exercise 6.1 (and solution); VGT Sec 6.2 (subgroups) and exercise 6.17 (and solution).

Write up solutions to the following questions. All answers could be completed by hand or via Overleaf.

1. Before attempting this question, review the two algorithms for expressing a group G of order n as a set of permutations in S_n in VGT Section 5.4.4 pg 83-86 and Slides 2.4 (Cayley's Theorem). The first algorithm uses the Cayley diagram and the other uses the multiplication table.

   A Cayley diagram and multiplication table for the dihedral group D_4 are shown below.



	e	r	r^2	r^3	f	rf	r^2f	r^3f
e	e	r	r^2	r^3	f	rf	r^2f	r^3f
r	r	r^2	r^3	e	rf	r^2f	r^3f	f
r^2	r^2	r^3	e	r	r^2f	r^3f	f	rf
r^3	r^3	e	r	r^2	r^3f	f	rf	r^2f
f	f	r^3f	r^2f	rf	e	r^3	r^2	r
rf	rf	f	r^3f	r^2f	r	e	r^3	r^2
r^2f	r^2f	rf	f	r^3f	r^2	r	e	r^3
r^3f	r^3f	r^2f	rf	f	r^3	r^2	r	e

   (a) Label the vertices of the Cayley diagram from the set $\{1, \ldots, 8\}$ and use this to construct a permutation group isomorphic to D_4, and sitting inside S_8.

   (b) Label the entries of the multiplication table from the set $\{1, \ldots, 8\}$ and use this to construct a permutation group isomorphic to D_4, and sitting inside S_8.

   (c) Are the two groups you got in Parts (a) and (b) the same? (The answer will depend on your choice of labeling.) If "yes", then repeat Part (a) with a different labeling to yield a different group. If "no", then repeat Part (a) with a different labeling to yield the group you got in Part (b).

2. Before attempting this question, read the algorithm given in VGT Exercise 6.22 and 6.23(a), and review Slides 3.1 (subgroups) and Slides 3.2 (cosets, including index of a subgroup).

   Find all subgroups of the following groups, and arrange them in a Hasse diagram, or subgroup lattice. Moreover, label each edge between $K \leq H$ with the index, $[H : K]$. [You may check the list of all subgroups for each group using Group Explorer.]

   (a) $C_{23} = \langle r \mid r^{23} = 1 \rangle$;

(b) $C_{24} = \langle r \mid r^{24} = 1 \rangle$;

(c) $\mathbb{Z}_3 \times \mathbb{Z}_3 = \{(a,b) \mid a,b \in \{0,1,2\}\}$;

(d) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(a,b,c) \mid a,b,c \in \{0,1\}\}$; (*Tip*: it's notationally easier to write elements as binary strings, e.g., 110 instead of $(1,1,0)$);

(e) $S_3 = \{e, (1\,2), (2\,3), (1\,3), (1\,2\,3), (1\,3\,2)\}$;

(f) $A_4 = \{e, (123), (132), (124), (142), (134), (143), (234), (243), (12)(34), (13)(24), (14)(23)\}$;

(g) $Q_8 = \langle i, j, k \mid i^2 = j^2 = k^2 = ijk = -1 \rangle$.

3. For each subgroup H of S_4 described below, and determine what well-known group it is isomorphic to. Prove your answers using either the Cayley diagram, group presentation, or other tools we have learned so far. [For some of these subgroups H, you might find it helpful to first write down all elements of H.]

   (a) $H = \langle (1\,2)\,,\,(3\,4) \rangle$;

   (b) $H = \langle (1\,2)\,(3\,4)\,,\,(1\,3)\,(2\,4) \rangle$;

   (c) $H = \langle (1\,2)\,,\,(2\,3) \rangle$;

   (d) $H = \langle (1\,2)\,,\,(1\,3\,2\,4) \rangle$;

   (e) $H = \langle (1\,2\,3)\,,\,(2\,3\,4) \rangle$.

4. Before you attempt this question, review Slides 3.2 (cosets).

   Let subgroup $H \leq G$ be a subgroup of G. Recall that, if $x \in G$, the set $xH := \{xh \mid h \in H\}$ is a *left coset* of H.

   (a) Prove that if $x \in H$, then $xH = H$.

   (b) Prove that if $b \in aH$, then $aH = bH$.

   (c) Let $x \in G$. Define a bijective map f from H to xH, and prove that f is a bijection. Conclude that all left cosets of H have the same size.

   (d) Conclude that G is partitioned by the left cosets of H, all of which are equal size.

5. Before you attempt this question, slides 3.2 (cosets and Lagrange's Theorem), read VGT Sec 6.5 (Lagrange's Theorem) and VGT exercise 6.14 (and solution).

   Prove that if $[G : H] = 2$, then $xH = Hx$ for all $x \in G$. [Use the results of the previous problem.]

6. Prove the following.

   (a) Let $\mathcal{H}$ is a collection of subgroups of G. Show that $\displaystyle\bigcap_{H_\alpha \in \mathcal{H}} H_\alpha$ is a subgroup of G. [That is, show that $xy \in \bigcap_{H_\alpha \in \mathcal{H}} H_\alpha$ for all $x, y \in \bigcap_{H_\alpha \in \mathcal{H}} H_\alpha$, show that every element $x \in \bigcap_{H_\alpha \in \mathcal{H}} H_\alpha$ has an inverse in $\bigcap_{H_\alpha \in \mathcal{H}} H_\alpha$.]

   (b) For any subset $S \subseteq G$, the subgroup generated by S is defined as

   $$\langle S \rangle := \{s_1^{e_1} s_2^{e_2} \cdots s_k^{e_k} \mid s_i \in S,\ e_i \in \{-1, 1\}\}\,.$$

   That is, $\langle S \rangle$ consists of all finite "words" that can be written using the elements in S and their inverses. Note that the s_i's need not be distinct. Prove that

   $$\langle S \rangle = \bigcap_{S \subseteq H_\alpha \leq G} H_\alpha\,,$$

   where the intersection is taken over all subgroups of G that contain S. [To prove that $A = B$, you need to show that that $A \subseteq B$ and $B \subseteq A$.]