Math3230 Abstract Algebra Worksheet (related to HW3)

i. Prove algebraically that if $g^2 = e$ for every element of a group G, then G must be abelian. Hint: Exercise 5.38

ii. First, review cycle notations (Slides 2.3 or VGT 5.4.1-5.4.2, Fig. 5.32).

Compute the product of the following permutations (reading from left to right). Your answer for each should be a single permutation written in cycle notation as a product of disjoint cycles.

a. $(1\ 3\ 2)\,(1\ 2\ 5\ 4)(1\ 5\ 3)$ in S_5;

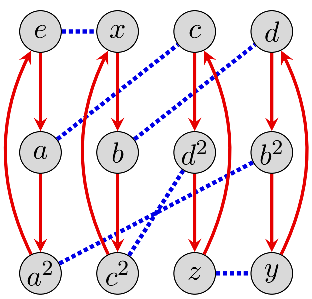
b. $(1\ 5)\,(1\ 2\ 4\ 6)\,(1\ 5\ 4\ 2\ 6\ 3)$ in S_6.

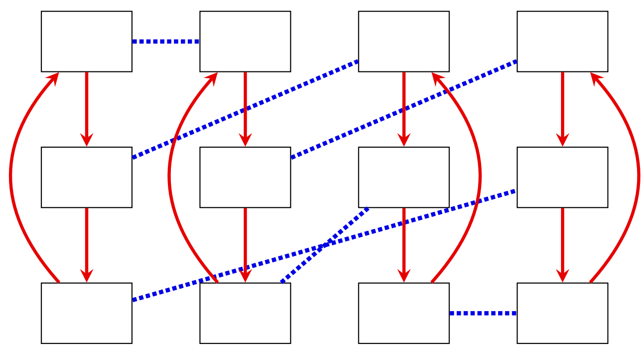
iii. (a) Make a Cayley diagram for the group G generated by the transpositions $a = (1\ 2)$ and $c = (3\ 4)$.

(b) Write down a group presentation for this Cayley diagram.

(c) Use either part (iiia) or (iiib) to describe a group you've seen which is isomorphic to G.

iv. The following Cayley diagram for A_4 labels the elements with letters instead of permutations: $A_4 = \{e, a, a^2, b, b^2, c, c^2, d, d^2, x, y, z\}$.





Redraw this Cayley diagram but label the nodes with the 12 even permutations from the previous problem. You need to determine which permutation corresponds to a, which to b, and so on.

Hint: Use the group presentation for A_4 from homework 2 to help you. There are many possible ways to do this. You should let a be one of the permutations of order 3, and let x be an element of order 2 (for example, $(12)(34)$) that satisfies the relations in the group presentation, then determine the remaining elements.

v. If σ is a cycle of odd length, prove that σ^2 is also a cycle.

Proof. Suppose σ is a cycle of odd length, so we can write $\sigma = (a_1, a_2, \ldots, a_{2k+1})$ in cycle notation. Then $\sigma^2 = (a_1, a_2, \ldots, a_{2k+1})(a_1, a_2, \ldots, a_{2k+1}) = (a_1$ you fill in the rest $\square$

Hint: To warm up, first compute $(1, 2, 3, 4, 5, 6)^2$ and $(1, 2, 3, 4, 5)^2$, written in cycle notation.

vi. Let τ be the cycle $(1, 2, 3, \ldots, k)$ in S_k written in cycle notation.
a. Prove that if σ is any permutation in S_k, then

$$\sigma^{-1}\tau\sigma = (\sigma(1), \sigma(2), \sigma(3), \ldots, \sigma(k))$$

Hint: Note that σ^{-1} maps the integer $\sigma(i)$ to $\sigma^{-1}(\sigma(i)) = i$. What does $\tau\sigma$ map i to?

b. Let $\mu = (b_1, b_2, \ldots, b_k)$ be a cycle of length k in S_k. Find a permutation $\sigma \in S_k$ such that $\sigma^{-1}\tau\sigma = \mu$.