Math3230 Abstract Algebra Homework 3

FIRST AND LAST NAME

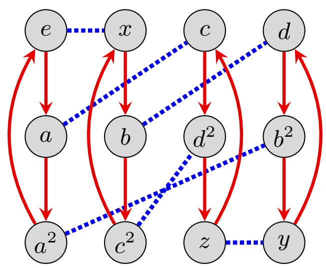
Write up the following. All answers could be completed by hand or via Overleaf.
Before you attempt to complete the questions, you should read and take notes of the following from *Visual Group Theory* (VGT): https://ebookcentral.proquest.com/lib/UCONN/detail.action?docID=3330331

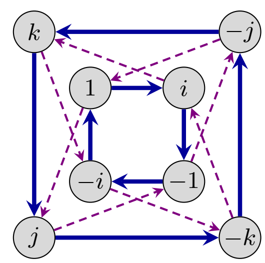
- Chapter 5 of VGT; Exercises (with answer key) of 5.6, 5.9, 5.20, 5.22, 5.24, 5.28, 5.30, 5.31, and 5.38.

1. Prove algebraically that if $g^2 = e$ for every element of a group G, then G must be abelian. Hint: Exercise 5.38

2. First, review the concept of orbits (Slides 1.5 or VGT Sec 5.1.3-5.1.4 pages 66-68).

   Carry out the following steps for the groups whose Cayley graphs are shown below.





   (a) Find the orbit of each element.
   (b) Draw the orbit graph of the group.

3. First, review cycle notations (Slides 2.3 or VGT 5.4.1-5.4.2, Fig. 5.32).

   Compute the product of the following permutations (reading from left to right). Your answer for each should be a single permutation written in cycle notation as a product of disjoint cycles.

   a. $(1\ 3\ 2)\,(1\ 2\ 5\ 4)(1\ 5\ 3)$ in S_5;
   b. $(1\ 5)\,(1\ 2\ 4\ 6)\,(1\ 5\ 4\ 2\ 6\ 3)$ in S_6.

4. a. Write out all $4! = 24$ permutations in S_4 in cycle notation as a product of disjoint cycles.
   b. Write each of the $4! = 24$ permutations above as a product of transpositions.
   c. Determine whether each of the $4! = 24$ permutations is even or odd.

5. First, review Slides 1.1 (Cayley diagram) and 1.3 (group presentations) or the videos linked next to them.

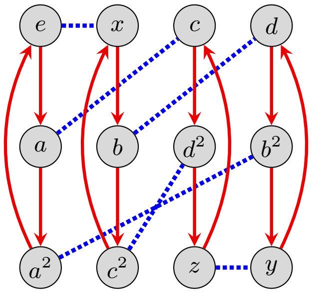
   The group S_3 can be generated by the transpositions $(1\ 2)$ and $(2\ 3)$. In fact, it has the following presentation

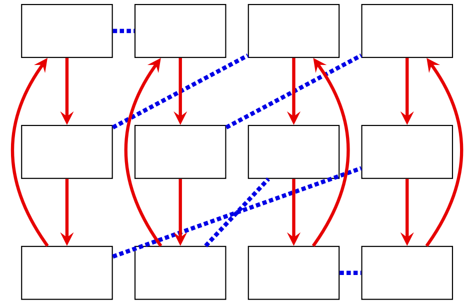
$$S_3 = \langle a, b \mid a^2 = e,\ b^2 = e\,,\ (ab)^3 = e\rangle,$$

   where one can take $a = (1\ 2)$ and $b = (2\ 3)$. Make a Cayley diagram for S_3 using this generating set.

6. (a) Make a Cayley diagram for the group G generated by the transpositions $a = (1\ 2)$ and $c = (3\ 4)$.
   (b) Write down a group presentation for this Cayley diagram.
   (c) Use either part (6a) or (6b) to describe a group you've seen which is isomorphic to G.

7. The following Cayley diagram for A_4 labels the elements with letters instead of permutations:





$A_4 = \{e, a, a^2, b, b^2, c, c^2, d, d^2, x, y, z\}$.

Redraw this Cayley diagram but label the nodes with the 12 even permutations from the previous problem. You need to determine which permutation corresponds to a, which to b, and so on.

Hint: Use the group presentation for A_4 from homework 2 to help you. There are many possible ways to do this. You should let a be one of the permutations of order 3, and let x be an element of order 2 (for example, $(12)(34)$) that satisfies the relations in the group presentation, then determine the remaining elements.

8. If σ is a cycle of odd length, prove that σ^2 is also a cycle.

Proof. Suppose σ is a cycle of odd length, so we can write $\sigma = (a_1, a_2, \ldots, a_{2k+1})$ in cycle notation. Then $\sigma^2 = (a_1, a_2, \ldots, a_{2k+1})(a_1, a_2, \ldots, a_{2k+1}) = (a_1$ you fill in the rest $\square$

Hint: To warm up, first compute $(1, 2, 3, 4, 5, 6)^2$ and $(1, 2, 3, 4, 5)^2$, written in cycle notation.

9. Let τ be the cycle $(1, 2, 3, \ldots, k)$ in S_k written in cycle notation.
a. Prove that if σ is any permutation in S_k, then

$$\sigma^{-1}\tau\sigma = (\sigma(1), \sigma(2), \sigma(3), \ldots, \sigma(k))$$

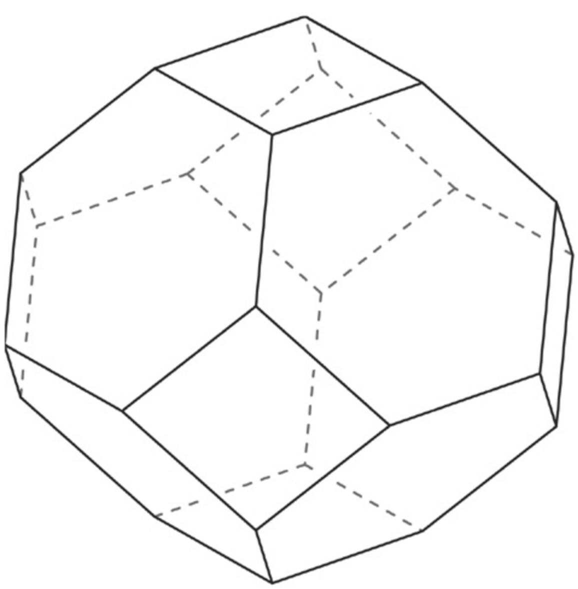
Hint: Note that σ^{-1} maps the integer $\sigma(i)$ to $\sigma^{-1}(\sigma(i)) = i$. What does $\tau\sigma$ map i to?
b. Let $\mu = (b_1, b_2, \ldots, b_k)$ be a cycle of length k in S_k. Find a permutation $\sigma \in S_k$ such that $\sigma^{-1}\tau\sigma = \mu$.

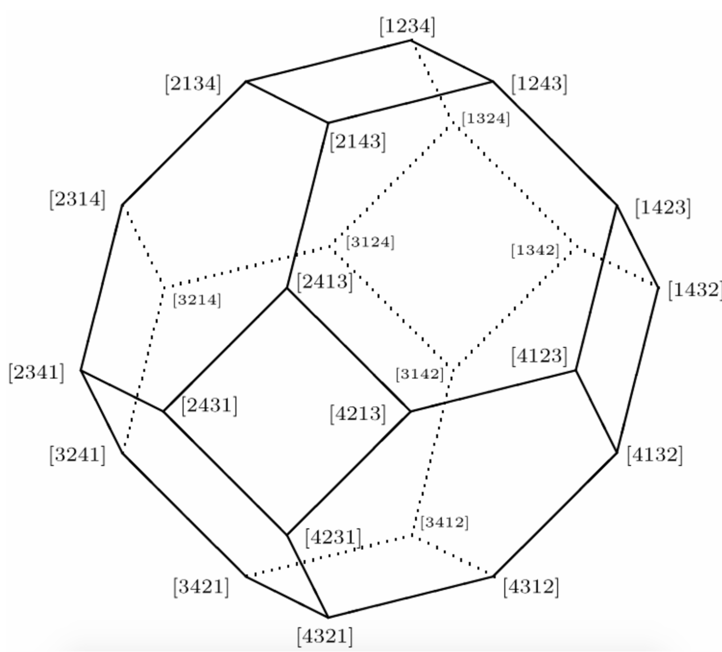
10. (a) The group S_4 can be generated by the transpositions (1 2), (2 3), and (3 4). Make a Cayley diagram for S_4 using this generating set. You can use or copy the unlabeled graph given below (on the left).

Instruction:

This Cayley diagram can be laid out on a polytope called a *permutahedron*, which is a truncated octahedron shown below. On the right, the vertices are labeled with the corresponding permutations in one-line notation.

Your task: Label the vertices on the unlabeled graph with the 24 permutations of S_4 *in cycle notation* (from question 4), then color the edges appropriately.





(b) Use this Cayley diagram to write down a group presentation for S_4. Hint: Other than the 2-cycle relations, there are only two types of closed paths, a diamond and a hexagon.