Abstract Algebra I
Practice Problems (Wed)

1. Suppose a group G of order 35 acts on a set S of size 9. Are there any fixed points? Prove your answer. [Hint: First, write down what sizes the orbits can be]

> **Solution:** Yes, there are at least two fixed points. Recall that an object $x \in S$ is a fixed point if and only if the orbit of x consists of only x.
>
> Proof: By the Orbit-Stabilizer Theorem, the size of the orbit must divide 35, so the size of an orbit can be 1, 5, or 7. So the possible orbit sizes are the following:
>
> 9 orbits, each of size 1.
>
> one orbit of size 5, and four orbits (each of size 1).
>
> one orbit of size 7, and two orbits (each of size 1).

2. Let H be a subgroup of G and let $x \in G$. You can use, without proof, a previous result (from egunawan.github.io/algebra/slides/sec3p3.pdf) that the set xHx^{-1} is a subgroup of G.

   (a) Prove that $H \cong xHx^{-1}$. [*Hint*: Define a map and show it is (i) a homomorphism, (ii) injective, and (iii) surjective.]

> **Solution:** Define $f : H \to xHx^{-1}$ by $f(h) = xhx^{-1}$.
> (i) To show that f is a homomorphism, let $a, b \in H$. Then
>
> $$\begin{aligned} f(ab) &= xabx^{-1} \\ &= xax^{-1}xbx^{-1} \\ &= f(a)f(b). \end{aligned}$$
>
> (ii) To show that f is injective, let $f(a) = f(b)$. Then $xax^{-1} = xbx^{-1}$.
> First multiply both sides on the right by x, then multiply both sides on the left by x^{-1}.
> (iii) To show that f is onto, let $y \in xHx^{-1}$. This means that $y = xhx^{-1}$ for some $h \in H$.
> Then $f(h) = xhx^{-1} = y$.

   (b) Prove that if H is the unique subgroup of G of (finite) order $|H|$, then H must be normal. You may use the result of Part (a) even if you didn't prove it.

> **Solution:** By Part (a), $xHx^{-1} \cong H$ for all $x \in G$, so $|H| = |xHx^{-1}|$ for all $x \in G$. Since H is the unique subgroup of order $|H|$, we must have $H = xHx^{-1}$ for all $x \in G$. So $H \lhd G$.

3. Answer the following questions about group actions.

(a) Suppose G acts on itself (so $S = G$) by conjugation. Then the orbit of any element $x \in G$ is ____________________________________. (Be as specific as possible!).

> **Solution:** See answer in Slides Sec 5.3 "Examples of group actions", Example 5.

(b) Let $G = S_4 = \langle (12), (23), (34) \rangle$ act on itself (so $S = G$) by conjugation, and let $\pi = (12)(34)$. Find $\text{Orb}(\pi)$ and $\text{Stab}(\pi)$. [*Hint*: First, use the orbit-stabilizer theorem to determine how big they are.]

> **Solution:** We know that $\text{Orb}(\pi)$ is the conjugacy class of π. The conjugacy class of π consists of the permutations which can be written as $(ab)(cd)$ in cycle notation. So $\boxed{\text{Orb}(\pi) = \{\pi, (13)(24), (14)(23)\}}$, which is of size 3.
>
> By the Orbit-Stabilizer Theorem, we know the order of $\text{Stab}(\pi)$ is $|S_4|/|\text{Orb}(\pi)| = 24/3 = 8$.
>
> We know of the elements in $\text{Stab}(\pi)$ is the identity permutation.
>
> <u>Possible method 1</u>: Draw the orbit of π as part of the action diagram using the given generating set for S_4. Then compute all possible compositions of arrows/loops starting at π and ending at π. You know to stop computing when you get 8 elements.
>
> <u>Possible method 2</u>: Note/recall that $p^{-1}(12)(34)p = (p(1)\ \ p(2))\ (p(3)\ p(4))$.
>
> So a permutation p is in $\text{Stab}(\pi)$ if and only if $(p(1)\ \ p(2))\ (p(3)\ \ p(4)) = (12)(34)$. For example, we can have $p(1) = 4, p(2) = 3$ and $p(3) = 1, p(4) = 2$, which means $p = (1\ 4\ 2\ 3)$.
>
> Using either method 1 or method 2, you should get that $\text{Stab}(\pi)$ consists of the following 8 permutations:
>
> e
>
> (12)
>
> (34)
>
> (12)(34)
>
> (13)(24)
>
> (14)(23)
>
> (1324)
>
> (1423)

4. Let $f\colon G \to H$ be a homomorphism, and let $K = \ker f$. Then, by a previous result, $K \lhd G$ (you don't need to prove).

   Prove the Fundamental Homomorphism Theorem: $G/K \cong \operatorname{Im} f$. Start by defining the map:

$$i\colon G/K \longrightarrow \operatorname{Im} f, \qquad i(gK) := f(g)\,.$$

   [You need to show that (i) i is well-defined, (ii) a homomorphism, (iii) 1–1, and (iv) onto. See egunawan.github.io/algebra/slides/notes/hw06iso1st.pdf]

Solution: To prove that i is well-defined, we need to show that if $aK = bK$ then $i(aK) = i(bK)$.

Suppose $aK = bK$. Then, for some $k \in K$, we have

$$ak = b. \tag{1}$$

Then

$$\begin{aligned}
i(aK) &= f(a) \\
&= f(a)\, e_H \\
&= f(a)f(k) \qquad \text{since } k \in K = \operatorname{Ker} f \\
&= f(ak) \qquad \text{since } f \text{ is a homomorphism} \\
&= f(b) \qquad \text{by (1)} \\
&= i(bK) \qquad \text{by definition of the map } i
\end{aligned}$$

Solution: To prove that i is a homomorphism, we need to show that $i(aK \cdot bK) = i(aK)i(bK)$. Recall that the binary operation for the quotient group is $aK \cdot bK := abK$.

Then

$$\begin{aligned}
i(aK \cdot bK) &= i(abK) \qquad \text{by definition of the binary operation of } G/K \\
&= f(ab) \qquad \text{by definition of } i \\
&= f(a)f(b) \qquad \text{since } f \text{ is a homomorphism} \\
&= i(aK)i(bK) \qquad \text{by definition of } i
\end{aligned}$$