1. Imagine that you have 1-cent coin and a 5-cent coin sitting side by side. At first, assume that both coins are showing heads (as opposed to tails), and the 1-cent coin is on the left. Consider the group generated by the following two actions.

   S: Switch left and right coins.

   f: Flip over the left coin.

   What familiar group is this isomorphic to? Prove your answer.

   Solution: Draw the Cayley diagram for $\langle S, f\rangle$ which turns out to have 8 elements and has the same structure as a Dihedral group.

2. For $n \geq 1$, let $[\pm n]$ denote the set $\{-n, -(n-1), \dots, -1\} \cup \{1, 2, \dots, n\}$. Let $S_{[\pm n]}$ denote the set of all bijections from $[\pm n]$ to $[\pm n]$.

   Review from Exam 2:

   - Define the set
   $$S_n^B := \{\text{bijections } w : [\pm n] \to [\pm n] \text{ where } w(-a) = -w(a)\},$$
   which is a subset of $S_{[\pm n]}$.
   For example, let $p = (2\ \ -4)$ be the function which swaps 2 and -4 and $p(j) = j$ for all other numbers j. Then $p \in S_{[\pm 4]}$, but $p \notin S_4^B$ because $p(-2) = -2 \neq -4 = -p(2)$.
   - Give an example of a non-identity function which is in S_4^B.

     Solution: For example, $p = (2\ \ -4)(-2\ 4)$ would work.

   - Prove that the set S_n^B is closed under function composition (using the above definition of the set).

     Solution: Suppose $h, g \in S_n^B$. Then
     $$\begin{aligned} f(g(-i)) &= f(-g(i)) \text{ since } g \in S_n^B \\ &= -f(g(i)) \text{ since } f \in S_n^B \end{aligned}$$

   (a) Define the set
   $$S_n^d := \{w \in S_n^B \mid \text{ the number of positive } i \text{ where } w(i) < 0 \text{ is even } \},$$
   which happens to be a subgroup of S_n^B.
   For example, below are all eight elements of S_2^B, but only some of them are in S_2^d.
   - The identity function is in S_2^d
   - The function $\mathbf{f} := (\mathbf{1},\ \mathbf{2})(-\mathbf{1},\ -\mathbf{2})$ swaps 1 and 2; and swaps -1 and -2.
   Note $f \in S_2^d$ because 0 positive numbers are sent to negative numbers.

- The function $\mathbf{g} := (\mathbf{1},\ -\mathbf{1})$ swaps 1 and -1, and it fixes 2 and -2.
  Note g is not in S_2^d because exactly one positive number is sent to a negative number.

- The function $\mathbf{gf} = (1,\ -1)\ (1,\ 2)(-1,\ -2) = (\mathbf{1},\ -\mathbf{2},\ -\mathbf{1},\ \mathbf{2})$ is a 4-cycle $1 \mapsto -2 \mapsto -1 \mapsto 2 \mapsto 1$.
  Note gf is not in S_2^d because exactly one positive number is sent to a negative number.

Determine whether the following functions (which are in S_2^B) are also in S_2^d.

- $\mathbf{gfg} = (\mathbf{1},\ -\mathbf{2})(-\mathbf{1},\ \mathbf{2})$
- $\mathbf{gfgf} = (\mathbf{1},\ -\mathbf{1})(\mathbf{2},\ -\mathbf{2})$
- $\mathbf{gfgfg} = (\mathbf{2},\ -\mathbf{2})$
- $\mathbf{gfgfgf} = (\mathbf{1},\ \mathbf{2},\ -\mathbf{1},\ -\mathbf{2})$

Solution:
See take-home questions for the final exam

(b) What familiar group is S_2^d isomorphic to? Prove your answer.

(c) Pick a minimal generating set for S_2^d and use it to draw a Cayley diagram for S_2^d. Clearly label each arrow and each vertex of the Cayley diagram.

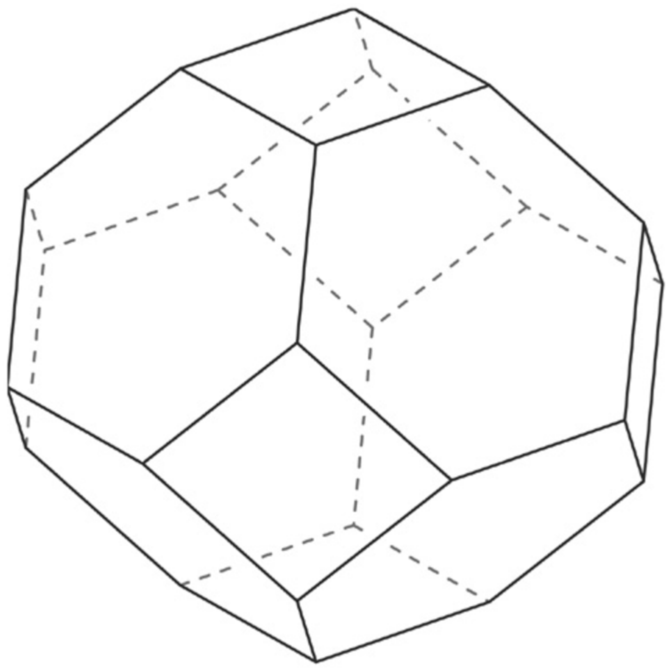
(d) Compute all conjugacy classes of S_2^B. That is, write down all $\mathrm{cl}_{S_2^B}(x) := \{pxp^{-1} \mid p \in S_2^B\}$. Recall that S_2^B is isomorphic to the 8-element dihedral group D_4
Compute all conjugacy classes of S_2^d.

3. (a) Use your computation to conjecture when elements in S_n^B are conjugate, for $n \geq 2$.
Conjecture when elements in S_n^d are conjugate, for $n \geq 3$.
(Hint: Similar to Thm 8. Attempt to prove your conjecture using Lemma 2)

(b) The group S_3^d has a minimal generating set with three generators
$a = (1\ 2)\ (-1\ \ -2),\quad b = (2\ 3)\ (-2\ \ -3),\quad c = (1\ \ -2)\ (-1\ 2)$
Make a Cayley diagram for S_3^d using a, b, c as generators. You can use the unlabeled graph given below.

(a) Your tasks:
- Label the bottom-most vertex with the identity function e.
- Label the left vertex connected to e with the function a.
- Label the right vertex connected to e with the function c.
- Label the top-most vertex with the function $abacba = cbacbc = (2\ \ -2)\ (3\ \ -3)$
- Color all edges appropriately, so that each edge corresponds to one of a, b, and c.
  Hint: The relation corresponding to the diamond shape means that two of the generators commute with each other. Determine whether $ab = ba$ or $ac = ca$ or $bc = cb$.
  Hint: A relation corresponding to a hexagon shape mimics the relation $xyx = yxy$ for the (Hexagon-shape) Cayley diagram for $D_3 \cong S_3$.

- There are 24 vertices total, but you only need to label the 12 "outer" vertices with the functions in S_3^d in cycle notation. Above, I've already told you where to put four of the functions, so you only need to label the remaining 8 vertices in cycle notation.



(c) Use this Cayley diagram to write down a group presentation for S_3^d. Hint: The relations are the doubled-sided arrows, diamonds and hexagons.

(d) If $n \geq 1$, give a formula for the order of S_n^B. Prove your answer.
If $n \geq 1$, give a formula for the order of S_n^d. Prove your answer.

Solution:
See take-home questions for the final exam

4. Consider the homomorphism $\phi \colon D_4 \to V_4$ determined by $\phi(r) = h$ and $\phi(f) = v$.

(a) Find the images of the remaining six elements of D_4.

(b) Is ϕ injective?

Solution: No, $\ker(\phi) = \{e, r^2\}$

(c) Find $\mathrm{Ker}(\phi)$.

(d) Find $\mathrm{Im}(\phi)$.

(e) What does the Fundamental/first Homomorphism Theorem tell us about ϕ? (Your answer should be more specific than just stating the FHT.)

Solution: $D_4/\langle r^2 \rangle$ is isomorphic to $\mathrm{Im}(\phi) = V_4$.

5. (a)
   - Define a map $f\colon$ Orb(x) to $G/Stab(x)$ which is a bijection. (See Lemma 1 in Slides 5.2: egunawan.github.io/algebra/slides/sec5p2.pdf)

   - **Carefully** prove Lemma 1. Divide into parts the proofs that (i) the map is well-defined, (ii) the map is injective, and (iii) the map is surjective, like in the slides.
     See all three proofs in Slides 5.2: egunawan.github.io/algebra/slides/sec5p2.pdf

   - For each of the examples of group action (Example 3: G acting on itself by right multiplication, Example 5: S_3 or D_6 acting on itself by conjugation, and Example 7: S_3 acting on cosets of $H = \langle(12)\rangle$ by right multiplication) in slides egunawan.github.io/algebra/slides/sec5p3.pdf, do the following:

     (i) pick an element $x \in S$;
     (ii) compute its orbit $\mathrm{Orb}(x)$;
     (iii) compute the subgroup $\mathrm{Stab}(x)$;
     (iv) use Lemma 1 to prove that $\mathrm{Orb}(x)$ is in bijection with $G/\mathrm{Stab}(x)$.

   (b)
   - Carefully state the Orbit-Stabilizer theorem (Theorem 1).
   - Use Lemma 1 to prove the Orbit-Stabilizer theorem.

6. (a) Prove that if $H < G$ and $[G : H] = 2$, then $H \lhd G$.

   Solution: The notation $[G : H] = 2$ means that there are exactly two left cosets of H. So the group G is partitioned into H and another left coset xH, hence, for all $x \notin H$, we have $xH = G - H$. Similarly, for all $x \notin H$, we have $Hx = G - H$.

   (b) Formally define the following terms: (i) an *action* of a group G on a set S, (ii) the orbit of $s \in S$, (iii) the stabilizer of $s \in S$.

   (c) Carefully state the orbit-stabilizer theorem.

   (d) Let G act on itself by conjugation, that is, $S = G$ and we have a homomorphism $\phi\colon G \to \mathrm{Perm}(S)$, where

   $$\phi(g) \quad \text{is} \quad \text{the permutation sending each } x \text{ to } g^{-1}xg.$$

   Carefully describe the orbits, the stabilizers, and the fixed points of this action.

   Solution:
   The orbits are the conjugacy classes, so, the orbit $\mathrm{Orb}(x)$ is $\mathrm{cl}_G(x)$, the conjugacy class of x.
   See answer in Slides 5.3: egunawan.github.io/algebra/slides/sec5p3.pdf

   (e) Prove that the size of any conjugacy class divides $|G|$.

   Solution: See answer in Slides 5.3: egunawan.github.io/algebra/slides/sec5p3.pdf

   (f)

   Definition 1. A nontrivial group G is called *simple* if its only normal subgroups are the trivial group and the group G itself.
   Show that if a group G contains an element $x \in G$ that has exactly two conjugates (itself included), then G is not simple. [*Hint*: You can use the previous parts of the problem!]

Solution: Suppose G contains an element which has exactly two conjugates, so $\mathrm{cl}_G(x)$ has size exactly 2.

Consider the group action described in part (d). Recall that $\mathrm{Stab}(x)$ is a subgroup of G. Note that $\mathrm{Stab}(x)$ is not the entire group G, otherwise $\mathrm{cl}_G(x)$ would be the singleton $\{x\}$.

We will now prove that $\mathrm{Stab}(x)$ is a normal subgroup of G.

Lemma 1 (for the Orbit-Stabilizer theorem) says that there is a bijection between $\mathrm{Orb}(x)$ and the set $G/\mathrm{Stab}(x)$, the set of right cosets of $\mathrm{Stab}(x)$. So there is a bijection between $\mathrm{cl}_G(x)$ and the set $G/\mathrm{Stab}(x)$.

Since $\mathrm{cl}_G(x)$ has size 2, the set $G/\mathrm{Stab}(x)$ also has size 2, which means $[G : \mathrm{Stab}(x)] = 2$. By part (a), $\mathrm{Stab}(x) \lhd G$.

Fact Sheet

Lemma 2. For any $p \in S_n$, we have $p^{-1}\ (a_1\ a_2\ \ldots\ a_k)\ p = (\,p(a_1)\ p(a_2)\ \ldots\ p(a_k)\,)$, reading left to right.

Definition 3. Let H be a subgroup of G. Recall that, if $x \in G$, the set

$$xH := \{xh \mid h \in H\}$$

is a *left coset* of H.

Theorem 4. Assume G is finite. If H is a subroup of G, then $|H|$ divides $|G|$.

Proof. Suppose there are n left cosets of the subgroup H. Since they are all the same size and they partition G, we must have $|G| = \underbrace{|H| + \cdots + |H|}_{n \text{ copies}} = n\,|H|$. ☐

Corollary 5. If G is a finite group and H is a subgroup of G, then $[G : H] = \frac{|G|}{|H|}$.

Theorem 6. Let H be a subgroup of G. Then the following are all equivalent.

(i) $gH = Hg$ for all $g \in G$; ("left cosets are right cosets");

(ii) $gHg^{-1} = H$ for all $g \in G$; ("only one conjugate subgroup")

(iii) $ghg^{-1} \in H$ for all $h \in H, g \in G$; ("closed under conjugation").

Corollary 7. If $[G : H] = 2$ then $H \lhd G$.

Theorem 8. Two permutations in S_n are conjugate if and only if they have the same cycle type.

Definition 9. A group *homomorphism* is a function $\phi\colon (G_1, *) \to (G_2, \circ)$ satisfying

$$\phi(a * b) = \phi(a) \circ \phi(b), \qquad \text{for all } a, b \in G_1\,.$$

Proposition 10. Let $f : G_1 \to G_2$ be a homomorphism of groups. Then

i. If e_1 is the identity of G_1, then $f(e_1)$ is the identity of G_2.

ii. For any element $g \in G_1$, $f(g^{-1}) = [f(g)]^{-1}$.

iii. If H_1 is a subgroup of G_1, then $f(H_1)$ is a subgroup of G_2.

iv. If H_2 is a subgroup of G_2, then $f^{-1}(H_2) = \{g \in G_1 : f(g) \in H_2\}$ is a subgroup of G_1.