Final Exam Questions (Take-Home)

Definitions

For $n \geq 1$, let $[\pm n]$ denote the set $\{-n, -(n-1), \dots, -1\} \cup \{1, 2, \dots, n\}$. Let $S_{[\pm n]}$ denote the set of all bijections from $[\pm n]$ to $[\pm n]$.
Review from Exam 2:

- Define the set
$$S_n^B := \{\text{bijections } w : [\pm n] \to [\pm n] \text{ where } w(-a) = -w(a)\},$$
which is a subset of $S_{[\pm n]}$. In fact, S_n^B is a subgroup of $S_{[\pm n]}$.

  For example, let $p = (2 \ -4)$ be the function which swaps 2 and -4 and $p(j) = j$ for all other numbers j. Then $p \in S_{[\pm 4]}$, but $p \notin S_4^B$ because $p(-2) = -2 \neq -4 = -p(2)$.

- Give an example of a non-identity function which is in S_4^B.

  Solution: The function $(2,\ -4)(-2,\ 4)$ and the function $(1,\ 2,\ -3,\ 4)(-1,\ -2,\ 3,\ -4)$ would both work

- Prove that the set S_n^B is closed under function composition (using the above definition of the set).

  Solution: Suppose $h, g \in S_n^B$. Then
$$\begin{aligned} f(g(-i)) &= f(-g(i)) \text{ since } g \in S_n^B \\ &= -f(g(i)) \text{ since } f \in S_n^B \end{aligned}$$

(a) Define the set
$$S_n^d := \{w \in S_n^B \mid \text{ the number of positive } i \text{ where } w(i) < 0 \text{ is even } \},$$
which happens to be a subgroup of S_n^B.

For example, below are all eight elements of S_2^B, but only some of them are in S_2^d.

1. The identity function is in S_2^d
2. The function $\mathbf{f} := (\mathbf{1},\ \mathbf{2})(-\mathbf{1},\ -\mathbf{2})$ swaps 1 and 2; and swaps -1 and -2.
   Note $f \in S_2^d$ because 0 positive numbers are sent to negative numbers.
3. The function $\mathbf{g} := (\mathbf{1},\ -\mathbf{1})$ swaps 1 and -1, and it fixes 2 and -2.
   Note g is not in S_2^d because exactly one positive number is sent to a negative number.
4. The function $\mathbf{gf} = (1,\ -1)\ (1,\ 2)(-1,\ -2) = (\mathbf{1},\ -\mathbf{2},\ -\mathbf{1},\ \mathbf{2})$ is a 4-cycle $1 \mapsto -2 \mapsto -1 \mapsto 2 \mapsto 1$.
   Note gf is not in S_2^d because exactly one positive number is sent to a negative number.
5. $\mathbf{gfg} = (\mathbf{1},\ -\mathbf{2})(-\mathbf{1},\ \mathbf{2})$
6. $\mathbf{gfgf} = (\mathbf{1},\ -\mathbf{1})(\mathbf{2},\ -\mathbf{2})$
7. $\mathbf{gfgfg} = (\mathbf{2},\ -\mathbf{2})$
8. $\mathbf{gfgfgf} = (\mathbf{1},\ \mathbf{2},\ -\mathbf{1},\ -\mathbf{2})$

Optional (for partial credit):

(a) Below are all eight elements of S_2^B. Circle all elements of S_2^d.

- e
- $(\mathbf{1},\ \mathbf{2})\,(-\mathbf{1},\ -\mathbf{2})$
- $(\mathbf{1},\ -\mathbf{1})$
- $(\mathbf{1},\ -\mathbf{2},\ -\mathbf{1},\ \mathbf{2})$
- $(\mathbf{1},\ -\mathbf{2})\,(-\mathbf{1},\ \mathbf{2})$
- $(\mathbf{1},\ -\mathbf{1})\,(\mathbf{2},\ -\mathbf{2})$
- $(\mathbf{2},\ -\mathbf{2})$
- $(\mathbf{1},\ \mathbf{2},\ -\mathbf{1},\ -\mathbf{2})$

(b) Write down three functions in S_5^d. Use either cycle notation or 2-line notation.

Take-home problem 1

(a) What familiar group is S_2^d isomorphic to? Prove your answer.

(b) Pick a minimal generating set for S_2^d and use it to draw a Cayley diagram for S_2^d. Clearly label each arrow and each vertex of the Cayley diagram.

(c) Write down all conjugacy classes of S_2^B. That is, write down all $\mathrm{cl}_{S_2^B}(x) := \{pxp^{-1} \mid p \in S_2^B\}$. Hint: S_2^B is isomorphic to the 8-element dihedral group D_4. Check the conjugacy classes of D_4: egunawan.github.io/algebra/slides/sec3p7.pdf

Write down all conjugacy classes of S_2^d.

(d) Use your computation to conjecture when elements in S_n^B are conjugate, for $n \geq 2$.

Conjecture when elements in S_n^d are conjugate, for $n \geq 3$.

(Hint: Similar to Thm 2. Attempt to prove your conjecture using Lemma 1)

(e) Write down textbooks or other sources you referenced.

If you worked on this during Friday class, write down the names of the people in your group.

Take-home problem 2

(a) The group S_3^d can be generated by functions

$a = (1\ 2)\ (-1\ \ -2), \quad b = (2\ 3)\ (-2\ \ -3), \quad c = (1\ \ -2)\ (-1\ 2)$

Make a Cayley diagram for S_3^d using a, b, c as generators. Use the unlabeled graph given in Figure 1.

Your tasks:

- Label the bottom-most vertex with the identity function e.
- Label the left vertex connected to e with the function a.
- Label the right vertex connected to e with the function c.
- (Therefore, the third vertex connected to e must be the function b.)
- Label the top-most vertex with the function $abacba = cbacbc = (2,\ -2)\ (3,\ -3)$
- Color all edges appropriately, so that each edge corresponds to one of a, b, and c.
  Hint: The relation corresponding to the diamond shape means that two of the generators commute with each other. Determine whether $ab = ba$ or $ac = ca$ or $bc = cb$.
  Hint: Each hexagon shape corresponds to a relation of the form $xyx = yxy$.
- There are 24 vertices total, but you only need to label the 12 "outer" vertices by the functions in S_3^d. Above, I've already told you where to put four of the functions, so you only need to label the remaining 8 vertices. Use cycle notation or 2-line notation.

(b) Use this Cayley diagram to write down a group presentation for S_3^d.

Hint: The relations come from the three types of (doubled-sided) arrows, one type of diamond, and two types of hexagons.

(c) (Optional, for partial credit) If $n \geq 1$, give a formula for the order of S_n. Prove your answer.

If $n \geq 1$, give a formula for the order of S_n^B. Prove your answer.

If $n \geq 1$, give a formula for the order of S_n^d. Prove your answer.

(Hint: If $n \geq 3$, the group S_n^B is *neither* a Dihedral group nor a symmetric group.)

(Hint: If $n \geq 4$, the group S_n^d is *neither* a Dihedral group nor a symmetric group.)

(d) Write down textbooks or other sources you referenced.

If you worked on this during Friday class, write down the names of the people in your group.

A few useful facts

Lemma 1. For any $p \in S_n$, we have $p^{-1}\ (a_1\ a_2\ \ldots\ a_k)\ p = (\,p(a_1)\ p(a_2)\ \ldots\ p(a_k)\,)$, reading left to right.

Theorem 2. Two permutations in S_n are conjugate if and only if they have the same cycle type.

For example, $(1\ 2\ 3\ 4)\ (5\ 6)$ and $(1\ 4\ 5\ 6)\ (2\ 7)$ are conjugate.

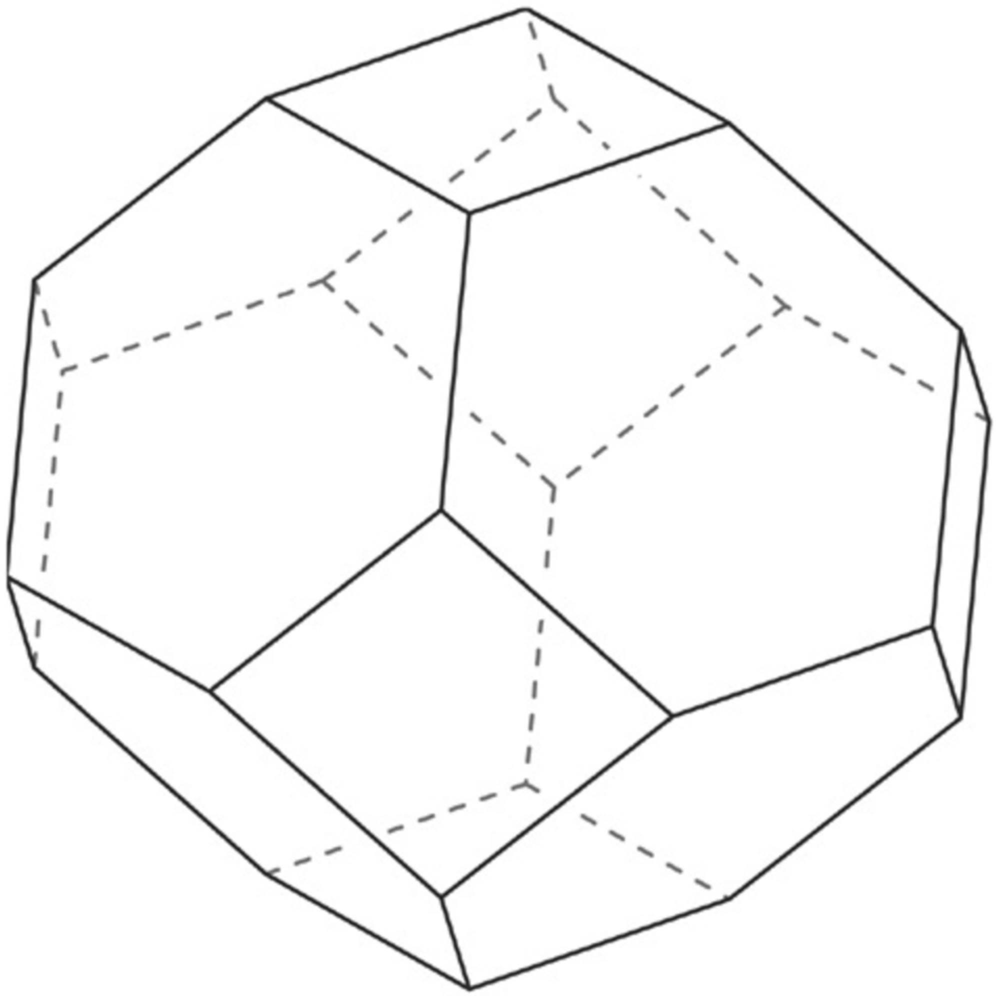

Figure 1: Unlabeled Cayley diagram of S_n^d