Contents

Study Suggestion: Pick a few questions from each section that look the most challenging.

1 Review cosets

1. (a) Let H be a subgroup of a group G, and let $x \in G$. Define a bijective map f from H to xH.

> **Solution:** Define
> $$f \colon H \longrightarrow xH\,, \qquad \text{by } f(h) = xh$$
> for all $h \in H$.

(b) Show that this map is surjective.

> **Solution:** Suppose $b \in xH$. Then by definition of left coset, $b = xh$ for some $h \in H$. Let $a := h$. Then $f(a) = xa = xh = b$, as needed.

(c) Suppose G is a non-abelian group of order 1000 and H is a subgroup of order 20. Let x be an element of G which is not in H. (i) How many elements are in the left coset xH? (ii) How many elements are in the right coset Hx?

> **Solution:** (i-ii)The size of every left coset (and every right coset) is the same as the size of H, so the answer is 20 for both questions.

How many left cosets of H are there?

> **Solution:** By the corollary of Lagrange's Theorem, there are $1000/20 = 50$ left cosets of H.

2 Related to Sec 3.3 normal subgroups

Theorem 1 (Theorem 3 from Slides 3.3)**.** Let H be a subgroup of G. Then the following are all equivalent.

(i) $gH = Hg$ for all $g \in G$; ("left cosets are right cosets");

(ii) $gHg^{-1} = H$ for all $g \in G$; ("only one conjugate subgroup")

(iii) $ghg^{-1} \in H$ for all $h \in H, g \in G$; ("closed under conjugation").

(iv) The subgroup H is called *normal* in G.

2. (a) Consider the subgroup $H = \{(1), (1,2)\}$ of S_3. Is H normal?

> **Solution:** No, you can check that $(123)H$ is not equal to $H(123)$.
>
> Another example that would work is $(13)H \neq H(13)$.
> A possibly faster way to determine this is to see that (13) and (23) are conjugate to (12) but they are not in H, hence failing part (iii) of the above theorem for being normal.

(b) Consider the subgroup $J = \{(1), (123), (132)\}$ of S_3. Is J normal?

> **Solution:** Yes, there is only other left coset of J (other than J itself), and there is only other other right coset of J (other than J), so they must be the same.
> This satisfies part (i) of the above theorem, Theorem 1, for being normal.

(c) Consider the subgroup $H = \langle(1234)\rangle$ of S_4. Is H normal?

> **Solution:** No. A possible proof: We know that every 4-cycle is conjugate to (1234), but not every 4-cycle is in $H = \{(1), (1234), (13)(24), (1432)\}$. For example, the 4-cycle (1324) is not in H.

(d) Let $n > 2$. Is A_n a normal subgroup of S_n?

> **Solution:** Yes. Proof: There are exactly two left cosets of A_n in S_n. So the left coset xA_n which is not equal to A_n must equal the right coset which is not equal to A_n.

(e) Consider a mystery subgroup K of $\mathbb{Z}_5 \times \mathbb{Z}_8$. Is K normal?

> **Solution:** Every subgroup of an abelian group is normal, so K is normal.

(f) Prove or disprove (with a counterexample): If $K \lhd H \lhd G$, then $K \lhd G$.

> **Solution:** This is false. See key to HW5.
>
> Note: The non-abelian group of order 6 (S_3 or D_3) is too small to produce this example because the maximum chain of distinct subgroups $\{e\} \leq H \leq D_3$, and $\{e\}$ is always normal. There is no non-abelian group of order 7, so try to find an example within a non-abelian group of order 8.
>
> A possible strategy: We've seen that every subgroup is normal in its normalizer. Find a subgroup K (which is not normal in G) and let H be the normalizer of K.
>
> Another strategy: For a simple example, choose K and H from the subgroups of D_4. The edge between each arrow (the index of H in K) is 2, so each subgroup K is normal in subgroup H whenever there is an edge between them.

3. Let H be a subgroup of G. Given two fixed elements $a, b \in G$, define the sets

$$aHbH := \{ah_1bh_2 \mid h_1, h_2 \in H\} \qquad \text{and} \qquad abH := \{abh \mid h \in H\}.$$

(a) Prove that *if* H is normal then $aHbH \subset abH$.

> **Solution:** To show $aHbH \subset abH$, let $h_1, h_2 \in H$. We need to show that ah_1bh_2 can be written as abh for some $h \in H$. Since H is normal in G, the left coset bH is equal to the right coset Hb. Hence we can write h_1b as bh_3 for some $h_3 \in H$, so $ah_1bh_2 = abh_3h_2$, which is in abH since $h_3h_2 \in H$.

(b) Prove that the statement is false if we remove the "normal" assumption. That is, give a specific G and H and $a, b \in G$ such that $aHbH$ is not a subset of abH.

> **Solution:** Possible proof: Let $G = D_3$, let $H = \langle f \rangle$. But $rfre = rfr = f$, which is in $rHrH$ but not in $r^2H = \{r^2, r^2f\}$, so $rHrH \neq r^2H$.
>
> Try to come up with a similar proof but using S_3.
>
> Possible scratch work (thought process):
> Let $G = D_3$ (because every group with order 5 or lower is abelian). To come up with a counterexample, I have to make sure to pick a non-normal subgroup H (since the statement is true if H is a normal subgroup), so I can pick one of the subgroups which is generated by exactly one reflection, $\langle f \rangle$ or $\langle rf \rangle$ or $\langle r^2 f \rangle$.
>
> I pick $H := \{e, f\}$. To come up with a counterexample, I have to make sure to pick $a, b \notin H$ (otherwise the statement would be true).
>
> First, I try $a = r$ and $b = r$, and I check whether $aHbH = abH$.
>
> I first compute abH because it is easier to compute (How do I know it's easier to compute? Because abH is a left coset of H and we have done a lot of practice computing left cosets, and also because from Definition **??**, we see abH has a simpler definition that the other set).
>
> Computing abH, I get $abH = r^2H = \{r^2, r^2f\}$.
>
> Now, I try to find an element in $aHbH = rHrH$ which is not in r^2H. Since H has only two elements, to compute all elements of $aHbH$ I just need to compute $aebe$, $aebf$, $afbe$, and $afbf$. But I see that the first two are in abH by Definition of abH, so I will only check the last two elements.
>
> I try $afbe = rfr = f$, which is not in abH. This example would be enough to show that $rHrH \neq rrH$.
>
> (You can also try $a = b = rf$, or $a = r$ and $b = rf$, and see what happens.)

(c) In class, we proved that multiplication of cosets of N is well-defined if N is a normal subgroup.
Give an example where "multiplication" of cosets is not well-defined. That is, give a group G and a subgroup H where $a_1H = a_2H$ and $b_1H = b_2H$ but $a_1b_1H \neq a_2b_2H$.

> **Solution:** You can use the same G and H as in the previous question. Just make sure your a_1, a_2, b_1, b_2 are not in H.
>
> Another possible example is the following:
> Consider the symmetric group S_3 and let $J := \langle (1\ 2) \rangle$.
> Then the three left cosets of J are:
>
> (a) $J = \{e, (1\ 2)\}$,
>
> (b) $(132)J = (13)J = (1\ 3), (1\ 3\ 2)\}$, and
>
> (c) $(1\ 2\ 3)J = (2\ 3)J = \{(2\ 3), (1\ 2\ 3)\}$.
>
> Take $a_1 := (132)$, $a_2 := (13)$,
> $b_1 := (123)$, and $b_2 := (23)$.
>
> Then $a_1b_1J = (132)(123)J = eJ = J$, but $a_2b_2J = (13)(23)J = (123)J \neq J$.

(d) Prove that if $aH = bH$ then $a^{-1}b \in H$. (Use only definition of left coset and the fact that G is a group. Do not "multiply" both sides by $a^{-1}H$.)

3 Related to Sec 3.4 direct products

4. Give two groups A and B, what is the definition of $A \times B$?
What is the binary operation on $A \times B$?
What is the identity element of $A \times B$?

Solution: $(1_A, 1_B)$, where 1_A is the identity element of A, and 1_B is the identity element of B.

If $(a, b) \in A \times B$, what is the inverse $(a, b)^{-1}$ equal to?

Solution: (a^{-1}, b^{-1})

If none of A and B is the trivial group, then $A \times B$ is guaranteed to have at least four normal subgroups. What are those four subgroups?

Solution: See Slide 9 of Slides Sec 3.4 egunawan.github.io/algebra/slides/sec3p4.pdf

5. (a) True or false? The order of the group D_n is the same as the order of the group $C_2 \times C_n$.

Solution: True, the order is $2n$ for both.

(b) True or false? The group D_n is isomorphic to the group $C_2 \times C_n$.

Solution: False. If $n \geq 3$, the Dihedral group D_n is non-abelian, but $C_2 \times C_n$ is.

(c) True or false? The group C_{14} is isomorphic to the group $C_2 \times C_7$.

Solution: True.
A possible proof: Note that $\mathbb{Z}_2 \times Z_7$ can be generated by the single element $(1, 1) \in \mathbb{Z}_2 \times \mathbb{Z}_7$ which has order $14 = lcm(2, 7)$, so it is a cyclic group of order 14.

(d) True or false? The group $\mathbb{Z}_{16}$ is isomorphic to the group $\mathbb{Z}_4 \times \mathbb{Z}_4$.

Solution: False. The group $\mathbb{Z}_{16}$ contains an element of order 16, that is, the number 1. Every element in the group $\mathbb{Z}_4$ has order 1, 2, or 4, so every element in the group $\mathbb{Z}_4 \times \mathbb{Z}_4$ also has order 1, 2, or 4.

(e) Is $\mathbb{Z}_{12}$ isomorphic to $\mathbb{Z}_2 \times Z_6$?

Solution: No. The group $\mathbb{Z}_2 \times Z_6$ has no element of order 12.

(f) Write $\mathbb{Z}_{12}$ as a nontrivial direct product.

Solution: $\mathbb{Z}_4 \times \mathbb{Z}_3$.

(g) i. Write down all the subgroups of $\mathbb{Z}_3 \times \mathbb{Z}_3$.

Solution: Check the list of subgroups in Group Explorer.

ii. Use your answer to show that $\mathbb{Z}_3 \times \mathbb{Z}_3$ is not the same group as $\mathbb{Z}_9$.

Solution: See Example 3.28 in http://abstract.ups.edu/aata/section-subgroups.html, which explains why $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is not the same group as $\mathbb{Z}/4\mathbb{Z}$.

4 Related to Sec 3.5 Quotient groups

6. Let H be a subgroup of G.

(a) What does the notation G/H mean?

Solution: The set of all left cosets of H in G, that is, $\{xH \mid x \in G\}$.

(b) When does the quotient G/H form a group?

Solution: When H is a normal subgroup of G.

(c) If G/N is a quotient group, what is the binary operation of the quotient group G/N?

Solution: $aN \cdot bN := abN$.

(d) Consider the symmetric group S_3 and a subgroup $H := \langle(1\ 2)\rangle$. Is $S_3/\langle(1\ 2)\rangle$ a quotient group? Prove your answer. If it is a quotient group, what is it isomorphic to?

Solution: No, $S_3/\langle(1\ 2)\rangle$ is not a quotient group.
A possible proof: $\langle(1\ 2)\rangle$ is not normal in S_3. The left coset $(123)\langle(1\ 2)\rangle = \{(23), (123)\}$ and the right coset $\langle(1\ 2)\rangle(123) = \{(13), (123)\}$ are not equal.
Another way to see that H is not normal is to recall that there are conjugates of (12) which are not in H, namely, (13) and (23).

(e) Consider the symmetric group S_3 and a subgroup $J := \langle(1\ 2\ 3)\rangle$. Is S_3/J a quotient group? Prove your answer. If it is a quotient group, what is it isomorphic to?

Solution: Yes, S_3/J is a quotient group because J is normal in S_3.

A possible proof: Since the order of S_3 is 6 and the order of J is 3, there are two left cosets of J. Hence the left coset of J (which is not J itself) must be equal to the right coset of J (which is not equal to J itself).

Another way to see that J is normal is to recall that all conjugates of a 3-cycle are also 3-cycles, and J contains all 3-cycles of S_3.

The quotient froup S_3/J is isomorphic to C_2 since there are two left cosets of J in S_3.

(f) Consider the subgroup $H = \langle(1234)\rangle$ of S_4. Is S_4/H a quotient group? Prove your answer. If it is a quotient group, what is it isomorphic to?

Solution: No, because H is not normal in S_4. A possible proof: We know that every 4-cycle is conjugate to (1234), but not every 4-cycle is in $H = \{(1), (1234), (13)(24), (1432)\}$. For example, the 4-cycle (1432) is not in H.

(g) Consider the symmetric group S_4 and a subgroup $J := \langle(1\ 2\ 3)\rangle$. Is S_4/J a quotient group? Prove your answer. If it is a quotient group, what is it isomorphic to?

> **Solution:** No, because J is not normal in S_4. A possible proof: We know that every 3-cycle is conjugate to (123), but not every 3-cycle is in J. For example, the 3-cycle (124) is not in J.

7. (a) List all normal subgroups N of D_4.

> **Solution:** The trivial subgroup $\{e\}$,
> the only normal subgroup of order 2, $\langle r^2\rangle$,
> all the subgroups of order 4: $\langle r\rangle$, $\langle r^2, f\rangle$, $\langle r^2, rf\rangle$, and
> D_4 itself.

(b) For each N above, what familiar group is D_4/N isomorphic to?

> **Solution:** The only one that we have to compute carefully is $D_4/\langle r^2\rangle$. We know that the number of cosets in $D_4/\langle r^2\rangle$ is 4, but there are two groups of order 4 (up to isomorphism), so let's list the cosets in $D_4/\langle r^2\rangle$:
> $\langle r^2\rangle$, $r\langle r^2\rangle$, $f\langle r^2\rangle$, and $rf\langle r^2\rangle$.
> By inspection, we see that each element (each coset) in $D_4/\langle r^2\rangle$ has order 2, so this quotient group must be isomorphic to V_4, and not to C_4.
>
> Final answer:
> $D_4/\{e\} \cong D_4$
> $D_4/\langle r^2\rangle \cong V_4$,
> For each subgroup H of order 4, we have $D_4/H \cong C_2$, and
> $D_4/D_4 \cong \{e\}$.

5 Related to Sec 3.6 normalizers

Definition 1. The set of elements in G that vote in favor of H's normality is called the *normalizer of H in G*, denoted $N_G(H)$. That is,

$$N_G(H) = \{g \in G : gH = Hg\} = \{g \in G : gHg^{-1} = H\}.$$

What is the smallest that $N_G(H)$ can be?

> **Solution:** H

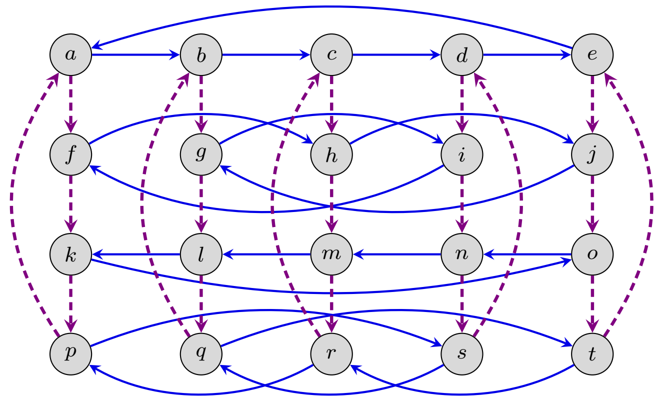
What is the largest it can be?

> **Solution:** G

When does the latter happens?

> **Solution:** $N_G(H) = G$ if and only if H is normal.

8. Let G be the group whose Cayley diagram is shown below, and suppose e is the identity element. Consider the subgroups $A = \langle a\rangle = \{a, b, c, d, e\}$ and $J = \langle j\rangle = \{e, j, o, t\}$.



Carry out the following steps for both of the subgroups A and J. List the cosets element-wise.

(a) Write G as a disjoint union of the subgroup's left cosets.

(b) Write G as a disjoint union of the subgroup's right cosets.

(c) Use your coset computation to immediately compute the normalizer of the subgroup. Based on the computation for the normalizer, what you can say about this subgroup?

> **Solution:** See answer key to HW 5.
> The normalizer for one of the subgroups is the entire G, meaning that this subgroup is normal. The normalizer for the other subgroup is the subgroup itself, meaning this group is as "unnormal" as possible.

(d) If G/A is a group, perform the quotient process and draw the resulting Cayley diagram for G/A. If G/J is a group, perform the quotient process and draw the resulting Cayley diagram for G/J.

6 Related to Sec 3.7 conjugation of an element, conjugacy classes

Let G be a group and $x \in G$. Review the definition of $\mathrm{cl}_G(x)$.

> **Solution:** $\mathrm{cl}_G(x) := \{gxg^{-1} \mid g \in G\}$

9. (a) Prove that two permutations $x, y \in S_n$ are conjugate if and only if they have the same cycle type.

> **Solution:** See the proof of Theorem 5 in notes Sec 3.7: egunawan.github.io/algebra/slides/notes/sec3p7whiteboard.pdf

(b) Prove that (12) and (14) in S_6 are conjugate by finding a permutation $p \in S_6$ such that $p^{-1}(12)p = (14)$.

> **Solution:** There are many possibilities for p. See the example after Lemma 4 in notes Sec 3.7: egunawan.github.io/algebra/slides/notes/sec

(c) List all permutations in S_4 which are conjugate to (1234). Use the fact from part (a).

> **Solution:** The answer is $(1234), (1432), (1243), (1342), (1324), (1423)$. Explanation: The permutations which are conjugate to (1234) in S_4 are all the 4-cycles.

10. Let G be a group and let Z be the set $\{z \in G \mid gz = zg \text{ for all } g \in G\}$. Prove that $\mathrm{cl}_G(x) = \{x\}$ if and only if $x \in Z$.

Solution: Slide 5 of egunawan.github.io/algebra/slides/sec3p7.pdf

11. Suppose N is a normal subgroup of G. Prove that if $x \in N$, then $\mathrm{cl}_G(x) \subset N$. (This means that every normal subgroup is the union of a collection of conjugacy classes).

Solution: Let $x \in N$. Since N is normal in G, we have $gxg^{-1} \in N$ for all $g \in G$. Thus, $\mathrm{cl}_G(x) := \{gxg^{-1} \mid g \in G\} \subset N$.

7 Related to center of a group

The *center* of a group G is the set

$$Z(G) = \{z \in G \mid gz = zg, \ \forall g \in G\} = \{z \in G \mid gzg^{-1} = z, \ \forall g \in G\}.$$

a. Prove that $Z(G)$ is normal in G by showing $ghg^{-1} \in H$ for all $h \in H, g \in G$; ("closed under conjugation").

Solution: Suppose $g \in G$. By part (iii) of Theorem 1, it is sufficient to show that $gzg^{-1} \in Z(G)$ for all $z \in Z(G)$. But, if $z \in Z(G)$, then $gzg^{-1} = z \in Z(G)$ for all $g \in G$.

b. Compute the center of the following groups: C_6, D_4, D_5, D_6, D_7 D_n.

Solution: C_6 is abelian, so the entire group is the center.
The center of D_4 is $\langle r^2 \rangle$. Reason: the half circle rotation commutes with every reflection (and every rotation). A different rotation does not commute with a reflection (for example, f). None of the reflections commutes with r.
The center of D_5 is the trivial group. Reason: None of the rotations commutes with f. None of the reflections commutes with r.

c. Compute the center of Q_8.
Recall that the elements of the Quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ are governed by the rules $i^2 = j^2 = k^2 = -1$, $ij = k, \quad jk = i, \quad ki = j, \qquad ji = -k, \quad kj = -i, \quad ik = -j\,.$

Solution: The only element (other than the identity 1) which commutes with every element in Q_8 is -1.

d. Consider the group A_n of even permutations, where $n > 3$. Prove that $(1\ 2\ 3)$ is not in the center of A_n by producing another even permutation which does not commute with $(1\ 2\ 3)$.

Solution: The element $(2\ 3\ 4)$ works. $(2\ 3\ 4)(1\ 2\ 3) = (12)(34)$
$(1\ 2\ 3)(2\ 3\ 4) = (13)(24)$

e. Consider the group A_n of even permutations, where $n > 3$. Prove that $(1\ 2)(3\ 4)$ is not in the center of A_n.

Solution: For example, you can show that the element $(1\ 2\ 3)$ does not commute with $(1\ 2)(3\ 4)$.

f. Compute the center of A_4
Hint: A non-identity permutation in S_4 is an even permutation if and only of its cycle notation is of the form $(ab)(cd)$ or (abc). (Make sure you can prove this!)
Do $(ab)(cd)$ and (abc) commute?

> **Solution:** Answer: The answer is the trivial group.
> Reason: The permutations (abc) and $(ab)(cd)$ do not commute.
> $(abc)(ab)(cd) = (a)(bdc)$ and $(ab)(cd)(abc) = (acd)(b)$.

g. Compute the center of S_4.
Hint: Every non-identity permutation in S_4 can be written in the form (ab), (abc), $(abcd)$, and $(ab)(cd)$. Can you find a permutation that does not commute with (ab)? With $(abcd)$?

h. Compute the center of S_2.

> **Solution:** This group is abelian, so the center is the entire group.

i. Compute the center of A_3.

> **Solution:** This group is abelian, so the center is the entire group. To see that A_3 is abelian, you can check that it can generated by the 3-cycle (123), so it is cyclic. Another way to see that A_3 is abelian, is to check its order, $3!/2$ which is equal to 3. We've seen that every group of order 3 is cyclic.

j. Prove or disprove that "the center of a direct product is the direct product of the centers", that is, $Z(A \times B) = Z(A) \times Z(B)$.

> **Solution:** True. First, it is clear that $Z(A \times B) \supset Z(A) \times Z(B)$.
> To show that $Z(A\times B) \subset Z(A)\times Z(B)$, let $(z_1, z_2) \in Z(A\times B)$. Then, by definition, $(z_1, z_2)(g_1, g_2) = (g_1, g_2)(z_1, z_2)$ for all $g_1 \in A$, $g_2 \in B$. This means that $(z_1g_1, z_2g_2) = (g_1z_1, g_2z_2)$ for all $g_1 \in A$, $g_2 \in B$. In other words, $z_1g_1 = g_1z_1$ and $z_2g_2) = g_2z_2$ for all $g_1 \in A$, $g_2 \in B$, so $z_1 \in Z(A)$ and $z_2 \in Z(B)$.

k. Use what you've done so far to compute the center of $D_n \times Q_8$. Draw the Cayley diagram for $Z(D_n \times Q_8)$.

> **Solution:** Take the cross product of the center of D_n and the center of Q_8. The Cayley diagram is either the one for $\mathbb{Z}_2$ or the one for V_4, that is, $\mathbb{Z}_2 \times \mathbb{Z}_2$.

8 Related to Sec 4.1 Homomorphisms and 4.2 Kernels

Proposition 1. Let $f : G_1 \to G_2$ be a homomorphism of groups. Then

(a) If e_1 is the identity of G_1, then $f(e_1)$ is the identity of G_2.
(b) For any element $g \in G_1$, $f(g^{-1}) = [f(g)]^{-1}$.
(c) If H_1 is a subgroup of G_1, then $f(H_1)$ is a subgroup of G_2.
(d) (i) If H_2 is a subgroup of G_2, then $f^{-1}(H_2) = \{g \in G_1 : f(g) \in H_2\}$ is a subgroup of G_1. (ii) Furthermore, if H_2 is normal in G_2, then $f^{-1}(H_2)$ is normal in G_1.

12. Prove all parts of Proposition 1.

> **Solution:** Proofs given under the Proposition 11.4 of Judson: http://abstract.ups.edu/aata/section-group-homomorphisms.html or slides Sec 4.1.

13. (a) Let $f : G_1 \to G_2$ be a homomorphism of groups. Prove that the kernel of f is a normal subgroup of G_1.

> **Solution:** Note that $\{e_2\}$ is a normal subgroup of the codomain G_2. By part (d)(ii) of above, $f^{-1}(\{e_2\})$ is normal.
>
> See also proof of Theorem 11.5 of Judson: http://abstract.ups.edu/aata/section-group-homomorphisms.html which is given in the paragraph between Proposition 11.4 and Theorem 11.5

(b) Let $f : G \to H$ be a group homomorphism. Show that f is injective if and only if the $\ker(f)$ is the trivial group $\{1_G\}$.

> **Solution:** For full proof, see hand-written notes Section 4.2 "Kernels": egunawan.github.io/algebra/slides/notes/sec4p2whiteboard.pdf

14. (a) Let $f : G_1 \to G_2$ be a *surjective* homomorphism. Prove that, if $N \lhd G_1$, then $f(N)$ is normal in G_2.

> **Solution:** We need to show that $x_2\, f(N)\, x_2^{-1} \subset f(N)$ for all $x_2 \in G_2$.
> Suppose $x_2 \in G_2$. Since f is surjective, there is $x_1 \in G_1$ such that $f(x_1) = x_2$. Note that every element in $f(N)$ can be written as $f(n)$ for some $n \in N$. Then
>
> $$\begin{aligned} x_2\, f(n)\, x_2^{-1} &= f(x_1)\, f(n)\, f(x_1)^{-1} \\ &= f(x_1)\, f(n)\, f(x_1^{-1}) \\ &= f(x_1\, n\, x_1^{-1}) \in f(N) \end{aligned}$$
>
> since $x_1\, n\, x_1^{-1} \in N$ (because N is normal in G_1).

(b) If $f : G_1 \to G_2$ is a homomorphism and N is a normal subgroup of G_1, is it possible that $f(N)$ is not normal in G_2? If so, give a counterexample.

> **Solution:** It is possible. Note that your counterexample would require a non-surjective hommormophism.
> For example, consider $f : C_2 = \{e, r\} \to S_3$ defined by $f(r) = (1\ 2)$ and let $N = C_2$. Then $f(N) = \langle (1\ 2) \rangle$, which is not normal in S_3.
> To see that $\langle (1\ 2) \rangle$ is not normal in S_3, check that the left coset and the right coset with coset representative $(1\ 2\ 3)$ are not equal.

15. I. Let $\phi : (\mathbb{Z}, +) \to (\mathbb{Z}, +)$ be the map given by $\phi(n) = 7n$ for $n \in \mathbb{Z}$. Find the kernel and the image of ϕ.

> **Solution:** The kernel of ϕ is the trivial subgroup $\{0\}$. The image of ϕ is $7\mathbb{Z}$, the subgroup of all integer multiples of 7.

II. Consider the group homomorphism $f : (\mathbb{R}.+) \to (\mathbb{C}^*, \times)$ defined by

$$f(\theta) = \cos\theta + i \sin\theta.$$

What is the kernel of f?
Give a bijective group homomorphism from the kernel of f to $(\mathbb{Z}, +)$. Prove that this map is a group homomorphism.

> **Solution:** From Example 11.7 in Judson Section 11.1: abstract.ups.edu/aata/section-group-homomorphisms.html
> Also discussed in class, Example 7 in Slides Sec 4.1 egunawan.github.io/algebra/slides/sec4p1.pdf

III. Let G be a group and let g be some element in G. Consider the group homomorphism from $\mathbb{Z}$ to G given by $f(n) = g^n$.

(a) If the order of g is infinite, what is the kernel of f? Justify.
(b) If the order of g is finite, say k, what is the kernel of f? Justify.

> **Solution:** From Example 11.9 in Judson Section 11.1: abstract.ups.edu/aata/section-group-homomorphisms.html
> Also discussed in class, Example 3 in Slides Sec 4.2 egunawan.github.io/algebra/slides/sec4p2.pdf

16. (a) Is there a homomorphism $f : (\mathbb{Z}_3, +) \to (\mathbb{Z}_4, +)$ where $f(1) = 1$? Prove your answer.

> **Solution:** No. See a possible proof in egunawan.github.io/algebra/slides/sec4p1.pdf

(b) True or false? Given two groups A and B, there exists a homomorphism from A to B. Prove your answer.

Solution: True, the map $f : A \to B$ where $f(x) = 1_B$ for all $g \in A$ (where 1_B is the identity element in B) is a homomorphism.

17. (a) Determine all possible homomorphisms from $(\mathbb{Z}_7, +) \to (\mathbb{Z}_{12}, +)$. Prove your answer.

Solution: Let f be such a homomorphism. By Prop 1 part 4 above, the kernel of f must be a subgroup of $\mathbb{Z}_7$. By Lagrange's Theorem, a subgroup of $\mathbb{Z}_7$ must be of order 1 or 7, so the only possible subgroups are $\{0\}$ and Z_7.

If $\ker(f) = \{0\}$, then f is injective by Slides 4.2 "Kernels". So $f(\mathbb{Z}_7)$ has order 7 (since each element of $\mathbb{Z}_7$ is sent to a unique element in $\mathbb{Z}_{12}$). By Prop 1 part 3 in Slides 4.1, the image of f is a subgroup of Z_{12}. But, again by Lagrange's Theorem, no subgroup of $\mathbb{Z}_{12}$ has order 7.

So $\ker(f) = \mathbb{Z}_7$.

Hence the only possible homomorphism $\mathbb{Z}_7 \to \mathbb{Z}_{12}$ is the zero map $f(a) = 0$ for all $a \in \mathbb{Z}_7$.

See also Example 11.8 in Judson Section 11.1: abstract.ups.edu/aata/section-group-homomorphisms.html

(b) Let $n \geq 2$. Determine all possible homomorphisms $(\mathbb{Z}_n, +) \to (\mathbb{Z}, +)$. Prove your answer.

Solution: Answer: the only homomorphism $\mathbb{Z}_n \to \mathbb{Z}$ is the zero homomorphism, that is, $\phi(g) = 0$ for all $g \in \mathbb{Z}_n$.

Proof: Suppose there is such a nonzero homomorphism $\phi\colon \mathbb{Z}/n \to \mathbb{Z}$. Then $\phi(1) = x$ for some nonzero $x \in \mathbb{Z}$ (otherwise, ϕ is the zero map).

Then we get

$$\phi(2) = \phi(1+1) = \phi(1) + \phi(1) = 2,$$
$$\phi(0) = \phi(1 + 1 + \cdots + 1) = \phi(1) + \phi(1) + \cdots + \phi(1) = n\,.$$

This is *impossible*, because $\phi(0) = 0$. (Identity is mapped to the identity.)

18. Given a homomorphism $f : G \to H$ define a relation $\sim$ on G by $a \sim b$ if $\phi(a) = \phi(b)$ for $a, b \in G$.

i. Show that this relation is an equivalence relation.

ii. Describe the equivalence classes. How many classes are there?

Solution: Check the three properties of being an equivalence class.

Description of the equivalence classes: Each element $h \in f(G)$ determines an equivalence class of the form $\{g \in G \mid f(g) = h\}$. The equivalence classes are in bijection to the elements of $f(G)$. There are as many equivalence classes as the number of elements in $f(G)$.

Extra information: If f is a surjection, then there is a bijection between the equivalence classes and H.

9 Related to Sec 4.3 First Isomorphism Theorem

19. (a) Consider the symmetric group S_3 and a (normal) subgroup $J := \langle(1\ 2\ 3)\rangle$. What familiar group is the quotient group S_3/J isomorphic to? Use the Fundamental Theorem of Homomorphism (1st Isomorphism Theorem) to formally prove your answer.

Solution: S_3/J is isomorphic to $C_2 = (\{1, -1\}, \cdot)$.

To prove this, define a map $f : S_3 \to C_2$ by sending $f(x) = 1$ if x is an even permutation and $f(x) = -1$ if x is an odd permutation.

We can check that f is a homomorphism by checking the possibilities of xy where x, y are both odd, both even, or of different parity.

To see that f is surjective, note that $f(e) = 1$ since $(12)(12) = e$, and $f((12)) = -1$.
We claim that $\ker(f) = J$. To see this, observe that all permutations in J are even, and all permutations not in J are odd.
By the 1st Isomorphism Theorem, $S_3/\ker(f)$ is isomorphic to C_2. Since $\ker(f) = J$, the result follows.

(b) Use the Fundamental Theorem of Homomorphism (1st Isomorphism Theorem) to formally prove that $\mathbb{Z}/n\mathbb{Z}$ is isomorphic to the cyclic group of order n, that is, $\mathbb{Z}_n$.

Solution: See Slide 10 of Slides Sec 4.3: egunawan.github.io/algebra/slides/sec4p3.pdfext or part 4 of my hand-written notes for the 1st Isomorphism Theorem: https://egunawan.github.io/algebra/slides/notes/hw06iso1st.pdf