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Study Suggestion: Pick a few questions from each section that look the most challenging.

1 Review cosets

1. (a) Let H be a subgroup of a group G, and let $x \in G$. Define a bijective map f from H to xH.
   (b) Show that this map is surjective.
   (c) Suppose G is a non-abelian group of order 1000 and H is a subgroup of order 20. Let x be an element of G which is not in H. (i) How many elements are in the left coset xH? (ii) How many elements are in the right coset Hx? How many left cosets of H are there?

2 Related to Sec 3.3 normal subgroups

Theorem 1 (Theorem 3 from Slides 3.3)**.** Let H be a subgroup of G. Then the following are all equivalent.

(i) $gH = Hg$ for all $g \in G$; ("left cosets are right cosets");
(ii) $gHg^{-1} = H$ for all $g \in G$; ("only one conjugate subgroup")
(iii) $ghg^{-1} \in H$ for all $h \in H, g \in G$; ("closed under conjugation").
(iv) The subgroup H is called *normal* in G.

2. (a) Consider the subgroup $H = \{(1), (1, 2)\}$ of S_3. Is H normal?
   (b) Consider the subgroup $J = \{(1), (123), (132)\}$ of S_3. Is J normal?
   (c) Consider the subgroup $H = \langle(1234)\rangle$ of S_4. Is H normal?
   (d) Let $n > 2$. Is A_n a normal subgroup of S_n?
   (e) Consider a mystery subgroup K of $\mathbb{Z}_5 \times \mathbb{Z}_8$. Is K normal?
   (f) Prove or disprove (with a counterexample): If $K \lhd H \lhd G$, then $K \lhd G$.
3. Let H be a subgroup of G. Given two fixed elements $a, b \in G$, define the sets

$$aHbH := \{ah_1bh_2 \mid h_1, h_2 \in H\} \qquad \text{and} \qquad abH := \{abh \mid h \in H\}.$$

   (a) Prove that *if* H is normal then $aHbH \subset abH$.
   (b) Prove that the statement is false if we remove the "normal" assumption. That is, give a specific G and H and $a, b \in G$ such that $aHbH$ is not a subset of abH.
   (c) In class, we proved that multiplication of cosets of N is well-defined if N is a normal subgroup. Give an example where "multiplication" of cosets is not well-defined. That is, give a group G and a subgroup H where $a_1H = a_2H$ and $b_1H = b_2H$ but $a_1b_1H \neq a_2b_2H$.
   (d) Prove that if $aH = bH$ then $a^{-1}b \in H$. (Use only definition of left coset and the fact that G is a group. Do not "multiply" both sides by $a^{-1}H$.)

3 Related to Sec 3.4 direct products

4. Give two groups A and B, what is the definition of $A \times B$?
   What is the binary operation on $A \times B$?
   What is the identity element of $A \times B$?
   If $(a, b) \in A \times B$, what is the inverse $(a, b)^{-1}$ equal to?
   If none of A and B is the trivial group, then $A \times B$ is guaranteed to have at least four normal subgroups. What are those four subgroups?
5. (a) True or false? The order of the group D_n is the same as the order of the group $C_2 \times C_n$.
   (b) True or false? The group D_n is isomorphic to the group $C_2 \times C_n$.
   (c) True or false? The group C_{14} is isomorphic to the group $C_2 \times C_7$.
   (d) True or false? The group $\mathbb{Z}_{16}$ is isomorphic to the group $\mathbb{Z}_4 \times \mathbb{Z}_4$.
   (e) Is $\mathbb{Z}_{12}$ isomorphic to $\mathbb{Z}_2 \times Z_6$?
   (f) Write $\mathbb{Z}_{12}$ as a nontrivial direct product.
   (g) i. Write down all the subgroups of $\mathbb{Z}_3 \times \mathbb{Z}_3$.
   ii. Use your answer to show that $\mathbb{Z}_3 \times \mathbb{Z}_3$ is not the same group as $\mathbb{Z}_9$.

4 Related to Sec 3.5 Quotient groups

6. Let H be a subgroup of G.
   (a) What does the notation G/H mean?
   (b) When does the quotient G/H form a group?
   (c) If G/N is a quotient group, what is the binary operation of the quotient group G/N?
   (d) Consider the symmetric group S_3 and a subgroup $H := \langle (1\ 2) \rangle$. Is $S_3/\langle (1\ 2) \rangle$ a quotient group? Prove your answer. If it is a quotient group, what is it isomorphic to?
   (e) Consider the symmetric group S_3 and a subgroup $J := \langle (1\ 2\ 3) \rangle$. Is S_3/J a quotient group? Prove your answer. If it is a quotient group, what is it isomorphic to?
   (f) Consider the subgroup $H = \langle (1234) \rangle$ of S_4. Is S_4/H a quotient group? Prove your answer. If it is a quotient group, what is it isomorphic to?
   (g) Consider the symmetric group S_4 and a subgroup $J := \langle (1\ 2\ 3) \rangle$. Is S_4/J a quotient group? Prove your answer. If it is a quotient group, what is it isomorphic to?
7. (a) List all normal subgroups N of D_4.
   (b) For each N above, what familiar group is D_4/N isomorphic to?

5 Related to Sec 3.6 normalizers

Definition 1. The set of elements in G that vote in favor of H's normality is called the *normalizer of H in G*, denoted $N_G(H)$. That is,

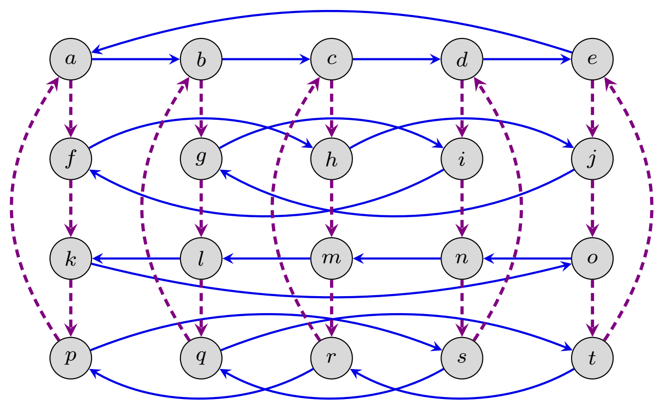
$$N_G(H) = \{g \in G : gH = Hg\} = \{g \in G : gHg^{-1} = H\}.$$

What is the smallest that $N_G(H)$ can be?

What is the largest it can be?

When does the latter happens?

8. Let G be the group whose Cayley diagram is shown below, and suppose e is the identity element. Consider the subgroups $A = \langle a \rangle = \{a, b, c, d, e\}$ and $J = \langle j \rangle = \{e, j, o, t\}$.



Carry out the following steps for both of the subgroups A and J. List the cosets element-wise.

(a) Write G as a disjoint union of the subgroup's left cosets.

(b) Write G as a disjoint union of the subgroup's right cosets.

(c) Use your coset computation to immediately compute the normalizer of the subgroup. Based on the computation for the normalizer, what you can say about this subgroup?

(d) If G/A is a group, perform the quotient process and draw the resulting Cayley diagram for G/A. If G/J is a group, perform the quotient process and draw the resulting Cayley diagram for G/J.

6 Related to Sec 3.7 conjugation of an element, conjugacy classes

Let G be a group and $x \in G$. Review the definition of $\mathrm{cl}_G(x)$.

9. (a) Prove that two permutations $x, y \in S_n$ are conjugate if and only if they have the same cycle type.
   (b) Prove that (12) and (14) in S_6 are conjugate by finding a permutation $p \in S_6$ such that $p^{-1}(12)p = (14)$.
   (c) List all permutations in S_4 which are conjugate to (1234). Use the fact from part (a).

10. Let G be a group and let Z be the set $\{z \in G \mid gz = zg \text{ for all } g \in G\}$. Prove that $\mathrm{cl}_G(x) = \{x\}$ if and only if $x \in Z$.

11. Suppose N is a normal subgroup of G. Prove that if $x \in N$, then $\mathrm{cl}_G(x) \subset N$. (This means that every normal subgroup is the union of a collection of conjugacy classes).

7 Related to center of a group

The *center* of a group G is the set

$$Z(G) = \{z \in G \mid gz = zg,\ \forall g \in G\} = \{z \in G \mid gzg^{-1} = z,\ \forall g \in G\}.$$

a. Prove that $Z(G)$ is normal in G by showing $ghg^{-1} \in H$ for all $h \in H, g \in G$; ("closed under conjugation").

b. Compute the center of the following groups: C_6, D_4, D_5, D_6, D_7 D_n.

c. Compute the center of Q_8.
   Recall that the elements of the Quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ are governed by the rules $i^2 = j^2 = k^2 = -1$, $ij = k, \quad jk = i, \quad ki = j, \qquad ji = -k, \quad kj = -i, \quad ik = -j$.

d. Consider the group A_n of even permutations, where $n > 3$. Prove that (1 2 3) is not in the center of A_n by producing another even permutation which does not commute with (1 2 3).

e. Consider the group A_n of even permutations, where $n > 3$. Prove that (1 2)(3 4) is not in the center of A_n.

f. Compute the center of A_4
   Hint: A non-identity permutation in S_4 is an even permutation if and only of its cycle notation is of the form $(ab)(cd)$ or (abc). (Make sure you can prove this!)
   Do $(ab)(cd)$ and (abc) commute?

g. Compute the center of S_4.
Hint: Every non-identity permutation in S_4 can be written in the form (ab), (abc), $(abcd)$, and $(ab)(cd)$. Can you find a permutation that does not commute with (ab)? With $(abcd)$?

h. Compute the center of S_2.

i. Compute the center of A_3.

j. Prove or disprove that "the center of a direct product is the direct product of the centers", that is, $Z(A \times B) = Z(A) \times Z(B)$.

k. Use what you've done so far to compute the center of $D_n \times Q_8$. Draw the Cayley diagram for $Z(D_n \times Q_8)$.

8 Related to Sec 4.1 Homomorphisms and 4.2 Kernels

Proposition 1. Let $f : G_1 \to G_2$ be a homomorphism of groups. Then

(a) If e_1 is the identity of G_1, then $f(e_1)$ is the identity of G_2.

(b) For any element $g \in G_1$, $f(g^{-1}) = [f(g)]^{-1}$.

(c) If H_1 is a subgroup of G_1, then $f(H_1)$ is a subgroup of G_2.

(d) (i) If H_2 is a subgroup of G_2, then $f^{-1}(H_2) = \{g \in G_1 : f(g) \in H_2\}$ is a subgroup of G_1. (ii) Furthermore, if H_2 is normal in G_2, then $f^{-1}(H_2)$ is normal in G_1.

12. Prove all parts of Proposition 1.

13. (a) Let $f : G_1 \to G_2$ be a homomorphism of groups. Prove that the kernel of f is a normal subgroup of G_1.
(b) Let $f : G \to H$ be a group homomorphism. Show that f is injective if and only if the $\ker(f)$ is the trivial group $\{1_G\}$.

14. (a) Let $f : G_1 \to G_2$ be a *surjective* homomorphism. Prove that, if $N \lhd G_1$, then $f(N)$ is normal in G_2.
(b) If $f : G_1 \to G_2$ is a homomorphism and N is a normal subgroup of G_1, is it possible that $f(N)$ is not normal in G_2? If so, give a counterexample.

15. I. Let $\phi : (\mathbb{Z}, +) \to (\mathbb{Z}, +)$ be the map given by $\phi(n) = 7n$ for $n \in \mathbb{Z}$. Find the kernel and the image of ϕ.
II. Consider the group homomorphism $f : (\mathbb{R}.+) \to (\mathbb{C}^*, \times)$ defined by

$$f(\theta) = \cos\theta + i \sin\theta.$$

What is the kernel of f?
Give a bijective group homomorphism from the kernel of f to $(\mathbb{Z}, +)$. Prove that this map is a group homomorphism.
III. Let G be a group and let g be some element in G. Consider the group homomorphism from $\mathbb{Z}$ to G given by $f(n) = g^n$.
(a) If the order of g is infinite, what is the kernel of f? Justify.
(b) If the order of g is finite, say k, what is the kernel of f? Justify.

16. (a) Is there a homomorphism $f : (\mathbb{Z}_3, +) \to (\mathbb{Z}_4, +)$ where $f(1) = 1$? Prove your answer.
(b) True or false? Given two groups A and B, there exists a homomorphism from A to B. Prove your answer.

17. (a) Determine all possible homomorphisms from $(\mathbb{Z}_7, +) \to (\mathbb{Z}_{12}, +)$. Prove your answer.
(b) Let $n \geq 2$. Determine all possible homomorphisms $(\mathbb{Z}_n, +) \to (\mathbb{Z}, +)$. Prove your answer.

18. Given a homomorphism $f : G \to H$ define a relation $\sim$ on G by $a \sim b$ if $\phi(a) = \phi(b)$ for $a, b \in G$.
i. Show that this relation is an equivalence relation.
ii. Describe the equivalence classes. How many classes are there?

9 Related to Sec 4.3 First Isomorphism Theorem

19. (a) Consider the symmetric group S_3 and a (normal) subgroup $J := \langle (1\ 2\ 3) \rangle$. What familiar group is the quotient group S_3/J isomorphic to? Use the Fundamental Theorem of Homomorphism (1st Isomorphism Theorem) to formally prove your answer.
(b) Use the Fundamental Theorem of Homomorphism (1st Isomorphism Theorem) to formally prove that $\mathbb{Z}/n\mathbb{Z}$ is isomorphic to the cyclic group of order n, that is, $\mathbb{Z}_n$.