1. (a) Let $n > 1$. What is the definition of an *even permutation* in S_n? (Use the definition you learned in class)

   Solution: An even permutation in S_n can be written as the product of an even number of transpositions (2-cycles).

   (b) For each permutation of S_6 below (written in cycle notation), draw a heart around it if it's an even permutation.

   $(1\,2)$ $(1\,2\,3)$ $(1\,2\,3)(4\,5)$ $(1\,2\,3)(4\,5\,6)$ $(1\,2\,3\,4)$ $(1\,2)(3\,4)$ $(1\,2)(3\,4)(5\,6)$ $(1\,2\,3\,4)(5\,6)$

   Solution: odd, even, odd, even, odd, even, odd, even

   (c) Let $n > 1$. Let A_n denote the set of even permutations in S_n.
   Circle the correct statement, and cross out the false statement:
   The left coset $(1\,2)A_n$ is equal to A_n /
   The left coset $(1\,2)A_n$ is not equal to A_n.
   Prove the correct statement.

   Solution: The left coset $(1\,2)A_n$ is not equal to A_n. For example, the identity element (1) is in A_n, so $(1\,2) = (1,\,2)(1)$ is in the left coset $(1\,2)A_n$, but $(1\,2)$ is not an even permutation, so $(1\,2) \notin A_n$.
   Remark: In fact, the left coset $(1\,2)A_n$ is the set of all odd permutations in S_n.

   (d) List all permutations in A_3 and S_3.

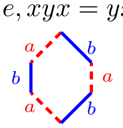
   Solution: $A_3 = \{(), (123), (132)\}$.
   $S_3 = \{(), (123), (132), (12), (23), (13)\}$

   (e) Pick a minimal generating set for A_3, and a minimal generating set for S_3. Hint: You have two choices for A_3 and 9 choices for S_3.

   Solution: For A_3, you can pick either $\{(1\,2\,3)\}$ or $\{(1\,3\,2)\}$.
   For S_3, you have 9 choices (since S_3 has the same structure as D_3 from homework 1 and 2). You can pick a set with two generators (both of order 2), like $\{(1\,2), (2\,3)\}$. You can also pick the set where one generator has order 2 and the other has order 3, like $\{(1\,2), (1\,2\,3)\}$

   (f) (i) Draw a Cayley diagram for these minimal generating sets - make sure to label each node with the corresponding permutations. If you have more than one generator, distinguish the different arrows by label or color/dashes. (ii) Write down the group presentations for these Cayley diagram.

   Solution: For A_3, a possible group presentation is $\langle x \mid x^3 = e \rangle$. The Cayley diagram looks like the Cayley diagram of a finite Cylic group: egunawan.github.io/algebra/slides/sec2p1.pdf
   For S_3, if both generators are of order 2, some possible group presentations are $\langle x, y \mid x^2 = y^2 = e, xyx = yxy \rangle$, $\langle x, y \mid x^2 = y^2 = (xy)^3 = e \rangle$. The abstract Cayley diagram would look like



For S_3, if one generator has order 2 and the other has order 3, some possible group presentations are $\langle r, f \mid r^3 = f^2 rfrf = e\rangle$. The Cayley diagram would look like the D_3 Cayley diagram in Slides 2.2: egunawan.github.io/algebra/slides/sec2p2.pdf

(g) If G is a finite group, the *index* $[G : H]$ of a subgroup $H \leq G$ is [give a definition, not a theorem!] ...

Solution: ... the number of left cosets of H.

(h) What is the index $[S_n : A_n]$ of the subgroup $A_n \leq S_n$? Justify this - you can use theorems you have seen in class (without reproving it here).

Solution: Answer = 2.

We have seen that there is a bijection between A_n and the set of odd permutations, so the number of A_n is half of $|S_n| = n!$. By the corollary of Lagrange's theorem, the number of left cosets of A_n is $|S_n|/|A_n|$.

(i) Let $G = \langle(1\,2\,3\,4\,5\,6\,7)\rangle$, the group generated by the permutation $(1\,2\,3\,4\,5\,6\,7)$, written in cycle notation. Prove that the only subgroups of G are the trivial group $\{()\}$ and G itself.

Solution: Proof: First, note that $(1\,2\,3\,4\,5\,6\,7)^7 = ()$ and $(1\,2\,3\,4\,5\,6\,7)^k \neq ()$ for any positive integer k less than 7, so the order of the permutation $(1\,2\,3\,4\,5\,6\,7)$ is 7. Hence G has order 7. By Lagrange's theorem, the order of any subgroup of G must be divisible by 7, so any subgroup of G has order 1 or 7.

2. For each statement below, determine if it is true or false. Prove your answer.

(a) If G is a non-abelian group, it must have a proper subgroup which is non-abelian.

Solution: False. The group S_3 is non abelian (for example $(12)(23) \neq (23)(12)$) but all its proper subgroups have order 1, 2, and 3, and you've see that all such groups are cyclic (hence abelian).

The group D_4 also works. All its proper subgroups have order 1, 2, and 4, and you've seen that all such groups are abelian.

(b) If the order of a group G is infinite (that is, if there are infinitely many elements in G), then the order of every $x \in G$ is also infinite. Recall that the order of x is the size of its orbit $\langle x\rangle$.

Solution: False. Consider the infinite Dihedral group $D_\infty = \langle r, f \mid f^2 = e, rfef = e\rangle$. There are infinitely many elements in D_∞ but the order of the element f is 2.

(c) There exists a dihedral group which is not abelian.

Solution: True. Proof: The dihedral group $D_2 = \langle r, f \mid r^2 = e, f^2 = e, rfrf = e\rangle = \langle r, f \mid r^2 = e, f^2 = e, rf = fr\rangle$ is abelian.

3. For each part below, compute the orbit of the element in the group. Your answer should be a list of elements from the group that ends with the identity.

   (a) The element r^2 in the group D_{10}

   Solution: $\{r^2, r^4, r^6, r^8, e\}$

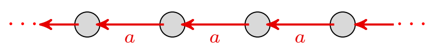
   (b) The element 12 in the group $C_{42} = \mathbb{Z}/42$

   Solution: $\{12, 24, 36, 6, 18, 30, 0\}$.

4. Recall that $\mathbb{Z}$ is a group under the operation of ordinary addition.

   (a) Create a Cayley diagram for it.

   Solution:



   (b) Is it abelian?

   Solution: Yes, it is a cyclic group, since it can be generated by the element 1 or -1.

   (c) Give a minimal generating set consisting of more than one element.

   Solution: For example, $\{2, 3\}$ or $\{7, 12\}$ would work.

5. (a) Find a group (of order larger than 1) such that there is only one solution to the equation $x^2 = e$, that is, the solution $x = e$, or explain why no such group exists.

   Solution: Yes. For example, the cyclic group of order 3. You can observe this from the multiplication table.

   (b) Find a group that has exactly two solutions to the equation $x^2 = e$, or explain why no such group exists.

   Solution: The cyclic group of order 4, $\langle r \mid r^4 = e \rangle$. The two solutions are $x = e$ and $x = r^2$.

   (c) Find a group that has more than two solutions to the equation $x^2 = e$, or explain why no such group exists.

   Solution: The rectangle puzzle (Klein-4 group) $\langle a, b \mid a^2 = b^2 = e, ab = ba \rangle$. There are four solutions, $x = e, x = a, x = b$, and $x = ab$. You can observe this from the multiplication table.

   (d) There are two non-isomorphic groups of order 6. What are their names? Specify which, if any, are abelian.

Solution: One is non-abelian, the Dihedral group D_3 or the symmetric group S_3. The other is the cyclic group C_6, which is abelian.

6. Answer the following questions about permutations and the symmetric group.

   (a) Write (1 2 3 4) as a product of *transpositions* (i.e., 2-cycles).

   Solution: $(12)(13)(14) = (23)(24)(12) = (34)(31)(23) = (14)(24)(34)$

   (b) What is the *inverse* of the element (1 3 2 6) (4 5) in S_6?

   Solution: $(45)(1623)$

   (c) The *order* of an element $g \in G$ is defined to be ...

   Solution: ... $|\langle g \rangle|$, the size of the orbit of g. Note that this (if finite) is also the minimum $k > 0$ such that $g^k = e$.

   (d) What is the order of the element (1 2 3 6) (4 5 7) in S_7?

   Solution: The order is 12 because $[(1\ 2\ 3\ 6)\,(4\ 5\ 7)]^i \neq id$ for $i = 1, 2, \ldots, 11$ and $[(1\ 2\ 3\ 6)\,(4\ 5\ 7)]^{12} = id$

   (e) Find an element of order 20 in S_9.

   Solution: (1 2 3 4 5) (6 7 8 9)