1. (a) Let $n > 1$. What is the definition of an *even permutation* in S_n? (Use the definition you learned in class)

   (b) For each permutation of S_6 below (written in cycle notation), draw a heart around it if it's an even permutation.
   $(1\,2)$ $(1\,2\,3)$ $(1\,2\,3)(4\,5)$ $(1\,2\,3)(4\,5\,6)$ $(1\,2\,3\,4)$ $(1\,2)(3\,4)$ $(1\,2)(3\,4)(5\,6)$ $(1\,2\,3\,4)(5\,6)$

   (c) Let $n > 1$. Let A_n denote the set of even permutations in S_n.
   Circle the correct statement, and cross out the false statement:
   The left coset $(1\,2)A_n$ is equal to A_n /
   The left coset $(1\,2)A_n$ is not equal to A_n.
   Prove the correct statement.

   (d) List all permutations in A_3 and S_3.

   (e) Pick a minimal generating set for A_3, and a minimal generating set for S_3. Hint: You have two choices for A_3 and 9 choices for S_3.

   (f) (i) Draw a Cayley diagram for these minimal generating sets - make sure to label each node with the corresponding permutations. If you have more than one generator, distinguish the different arrows by label or color/dashes. (ii) Write down the group presentations for these Cayley diagram.

   (g) If G is a finite group, the *index* $[G : H]$ of a subgroup $H \leq G$ is [give a definition, not a theorem!] …

   (h) What is the index $[S_n : A_n]$ of the subgroup $A_n \leq S_n$? Justify this - you can use theorems you have seen in class (without reproving it here).

   (i) Let $G = \langle(1\,2\,3\,4\,5\,6\,7)\rangle$, the group generated by the permutation $(1\,2\,3\,4\,5\,6\,7)$, written in cycle notation. Prove that the only subgroups of G are the trivial group $\{()\}$ and G itself.

2. For each statement below, determine if it is true or false. Prove your answer.

   (a) If G is a non-abelian group, it must have a proper subgroup which is non-abelian.

   (b) If the order of a group G is infinite (that is, if there are infinitely many elements in G), then the order of every $x \in G$ is also infinite. Recall that the order of x is the size of its orbit $\langle x \rangle$.

   (c) There exists a dihedral group which is not abelian.

3. For each part below, compute the orbit of the element in the group. Your answer should be a list of elements from the group that ends with the identity.

   (a) The element r^2 in the group D_{10}

   (b) The element 12 in the group $C_{42} = \mathbb{Z}/42$

4. Recall that $\mathbb{Z}$ is a group under the operation of ordinary addition.

   (a) Create a Cayley diagram for it.

   (b) Is it abelian?

   (c) Give a minimal generating set consisting of more than one element.

5. (a) Find a group (of order larger than 1) such that there is only one solution to the equation $x^2 = e$, that is, the solution $x = e$, or explain why no such group exists.

   (b) Find a group that has exactly two solutions to the equation $x^2 = e$, or explain why no such group exists.

   (c) Find a group that has more than two solutions to the equation $x^2 = e$, or explain why no such group exists.

   (d) There are two non-isomorphic groups of order 6. What are their names? Specify which, if any, are abelian.

6. Answer the following questions about permutations and the symmetric group.

   (a) Write (1 2 3 4) as a product of *transpositions* (i.e., 2-cycles).

   (b) What is the *inverse* of the element (1 3 2 6) (4 5) in S_6?

   (c) The *order* of an element $g \in G$ is defined to be ...

   (d) What is the order of the element (1 2 3 6) (4 5 7) in S_7?

   (e) Find an element of order 20 in S_9.