1. (a) Let $n > 1$. Let A_n and B_n denote the set of even permutations and the set of odd permutations, respectively. Define a map $f : A_n \to B_n$ by $f(\pi) = (1\,2)\pi$ for all $\pi \in A_n$. We proved that f is surjective (see notes 2.3 Symmetric groups and alternating groups).
Prove that this map is injective.

> **Solution:** Use the same proof as the proof of Proposition 5.17 here:
> abstract.ups.edu/aata/section-permutation-definitions.html#tlD

(b) Let H be a subgroup of a group G, and let $x \in G$. Define a bijective map f from H to xH.

> **Solution:** Define
> $$f\colon H \longrightarrow xH\,, \qquad \text{by } f(h) = xh$$
> for all $h \in H$.

(c) Show that this map is surjective.

> **Solution:** Suppose $b \in xH$. Then by definition of left coset, $b = xh$ for some $h \in H$. Let $a := h$. Then $f(a) = xa = xh = b$, as needed.

(d) Suppose G is a non-abelian group of order 1000 and H is a subgroup of order 20. Let x be an element of G which is not in H. (i) How many elements are in the left coset xH? (ii) How many elements are in the right coset Hx?

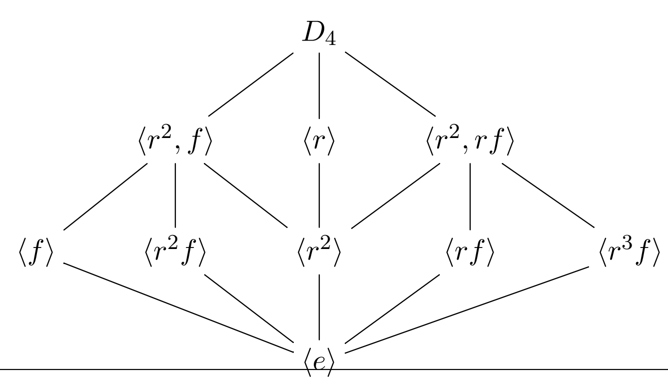
> **Solution:** (i-ii)The size of every left coset (and also right coset) is the same as the size of H, so the answer is 20 for both questions.

How many left cosets of H are there?

> **Solution:** By the corollary of Lagrange's Theorem, there are $1000/20 = 50$ left cosets of H.

2. a. Find all subgroups of D_4, and arrange them in a Hasse diagram, or subgroup lattice. Moreover, label each edge between $K \leq H$ with the index, $[H : K]$.

> **Solution:** The subgroup lattice of D_4 is shown below. The label on each edge is 2.



b. Is $f\langle r\rangle = \langle r\rangle f$? What about other left and right cosets of $\langle r\rangle$? Prove your answer.

Solution: Yes, $x\langle r\rangle = \langle r\rangle x$ for all $x \in D_3$. First, we see that the group $\langle r\rangle$ has order 4. We know that the group D_4 has order 8. By the Corollary of Lagrange's theorem, we get that $[D_4 : \langle r\rangle] = 4/2 = 2$. We've seen in class (Slides 3.2 on cosets) that this implies that the left cosets of $\langle r\rangle$ and the right cosets of $\langle r\rangle$ coincide.

c. Is the left coset $r^3 f\langle r^2, f\rangle$ equal to the right coset $\langle r^2, f\rangle r^3 f$?

Solution: Yes. Same explanation as the previous part.

3. For each statement below, determine if it is true or false. Prove your answer.

(a) If the order of a group G is infinite (that is, if there are infinitely many elements in G), then the order of every $x \in G$ is also infinite. Recall that the order of x is the size of its orbit $\langle x\rangle$.

Solution: False. Consider the infinite Dihedral group $D_\infty\langle r, f \mid f^2 = e, rfef = e\rangle$. There are infinitely many elements in D_∞ but the order of the element f is 2.

(b) Every cyclic group is abelian.

Solution: True. Proof: A cyclic group G is a group which can be generated by only one element, so $G = \langle r\rangle$ for some $r \in G$. If $x, y \in G$, then $x = r^k$ and $y = r^\ell$ for some $k, \ell \in \mathbb{Z}$. So $xy = r^k r^\ell = r^{k+\ell} = r^\ell r^k = yx$.

(c) Every abelian group is cyclic.

Solution: False. Proof: For example, the rectangle puzzle (or light-switch) group V_4 is not cyclic. It requires at least two generators.

(d) Every dihedral group is abelian.

Solution: False. Proof: The dihedral group D_3 of order 6 is not abelian, for example, rotation by 120^o followed by a flip is not the same as the same flip followed by a rotation by 120^o.

(e) Every dihedral group is not abelian.

Solution: False. Proof: The dihedral group $D_2 = \langle r, f \mid r^2 = e, f^2 = e, rfrf = e\rangle = \langle r, f \mid r^2 = e, f^2 = e, rf = fr\rangle$ is abelian.

(f) There is a cyclic group of order 100.

Solution: True. Proof: Take $\langle r \mid r^{100} = e \rangle$. Note that this group corresponds to the rigid rotations of a pentagon with 100 sides.

(g) There is a symmetric group of order 100

Solution: False. Proof: The number 100 is not equal to any factorial. Check that $4! = 24 < 100 < 5! = 120$.

(h) If some pair of non-identity elements in a group commute, then the group is abelian.

Solution: False. In D_3, the elements r and r^3 commute, but D_3 is not abelian.

(i) If every pair of elements in a group commute, the group is cyclic.

Solution: False. The group V_4 is not cyclic, but every pair of elements commutes.

(j) If every pair of elements in a group commute, the group is abelian.

Solution: True, by definition.

4. (a) Is there a dihedral group of order 27?

Solution: No. A dihedral group has n reflections and n rotations (for some positive integer n), so the order of a dihedral group is even.

(b) If an alternating group A_n has order M, what order does the symmetric group S_n have?

Solution: The order of S_n is $2M$, since we've seen that there is a bijection between the set of even permutations and the set of odd permutations and even permutations of S_n.

5. For each part below, compute the orbit of the element in the group. Your answer should be a list of elements from the group that ends with the identity.

(a) The element r^2 in the group D_{10}

Solution: $\{r^2, r^4, r^6, r^8, e\}$

(b) The element 10 in the cyclic group $C_{16} = \mathbb{Z}/16$

Solution: $\{10, 4, 14, 8, 2, 12, 6, 0\}$

(c) The element 25 in the group $C_{30} = \mathbb{Z}/30$

Solution: $\{25, 20, 15, 10, 5, 0\}$

(d) The element 12 in the group $C_{42} = \mathbb{Z}/42$

Solution: $\{12, 24, 36, 6, 18, 30, 0\}$.

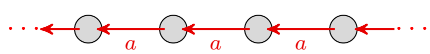
6. Open Group Explorer and sort the group library by order, with the smallest groups on top. Find a group which is not in any of the families of abelian/dihedral/symmetric/alternating groups, and explain why it's not in any of these families.

Solution: The group Q_4 (which has order 8) would work.
The group is not abelian, for example, $ij \neq ji$.
It is not a dihedral group D_4 because it only contains one element of order 2, while the dihedral group D_4 has four elements of order 2, namely, the four reflections.
It is not a symmetric or alternating group, since 8 is not equal to any factorial and 16 is also not equal to any factorial.

7. Recall that $\mathbb{Z}$ is a group under the operation of ordinary addition.

(a) Create a Cayley diagram for it.

Solution:



(b) Is it abelian?

Solution: Yes, it is a cyclic group, since it can be generated by the element 1 or -1.

(c) Give a minimal generating set consisting of more than one element.

Solution: For example, $\{2, 3\}$ or $\{7, 12\}$ would work.

(d) Give an object whose symmetries form $(\mathbb{Z}, +)$.

Solution: The frieze pattern labeled (1) or (4) in HW2 is an object whose symmetries the group $(\mathbb{Z}, +)$ describes.

8. (a) Is there a group (of order larger than 1) in which no element (other than the identity) is its own inverse?

Solution: Yes. For example, the cyclic group of order 3. You can observe this from the multiplication table.

(b) Find a group (of order larger than 1) such that there is only one solution to the equation $x^2 = e$, that is, the solution $x = e$, or explain why no such group exists.

Solution: The same group in the solution to previous part would work.

(c) Find a group that has exactly two solutions to the equation $x^2 = e$, or explain why no such group exists.

Solution: The cyclic group of order 4, $\langle r \mid r^4 = e \rangle$. The two solutions are $x = e$ and $x = r^2$.

(d) Find a group that has more than two solutions to the equation $x^2 = e$, or explain why no such group exists.

Solution: The rectangle puzzle (Klein-4 group) $\langle a, b \mid a^2 = b^2 = e, ab = ba \rangle$. There are four solutions, $x = e, x = a, x = b$, and $x = ab$. You can observe this from the multiplication table.

(e) Find a group with at least two elements in it, and only one solution to the equation $x^3 = e$ (that is, the solution $x = e$) or explain why no such group exists.

Solution: The groups C_2, C_4, and V_4 would work.

(f) Find a group that has more than two solutions to the equation $x^3 = e$, or explain why no such group exists.

Solution: In the cyclic group C_3, every element satisfies the equation $x^3 = e$.

(g) There are two non-isomorphic groups of order 6. What are their names? Specify which, if any, are abelian.

Solution: One is non-abelian, the Dihedral group D_3 which is isomorphic to the symmetric group S_3. The other is the cyclic group C_6, which is abelian.

(h) Suppose m is a positive integer. If there exists only one group of order m, to what family must this group belong? Why?

Solution: There is a cylic group $C_m = \langle r \mid r^m = e \rangle$ for every positive integer m, so this group must belong to the family of cylic groups.

9. (a) If H is a subgroup of G and $a \in G$, then a left coset aH is ... [give the definition]

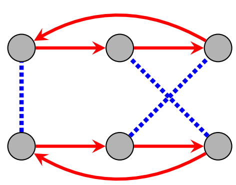
Solution: the set $\{ah \mid h \in H\}$

(b) If G is a finite group, the *index* $[G : H]$ of a subgroup $H \leq G$ is [give a definition, not a theorem!] ...

Solution: ... the number of left cosets of H.

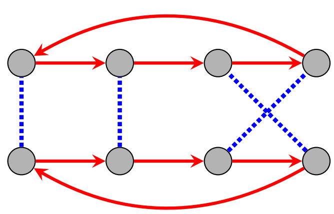
10. Determine whether each of the following diagrams are Cayley diagrams. If the answer is "yes," say what familiar group it represents, including the generating set. If the answer is "no," explain why.

(a)



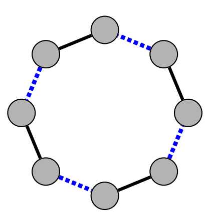
Solution: Yes. This is the Cayley diagram with group presentation $\langle r, f \mid r^3 = e, f^2 = e, rfrf = e\rangle$. These three patterns (relations) are regular in this graph.

(b)

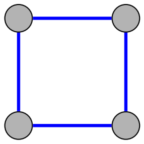


Solution: No. This graph is not regular. For example, we see a pattern red, blue equals blue, red (on the left side) but we see that red, blue does not equal blue, red (on the right side).

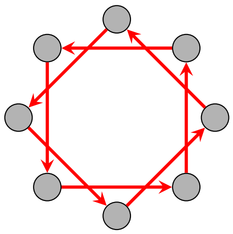
(c)



Solution: Yes. This is the Cayley diagram with group presentation $\langle a, b \mid a^2 = e, b^2 = e, (ab)^4 = e\rangle$ which gives us the Dihedral group D_4. The three patterns (relations) are regular in this graph.

(d) 

Solution: No. There is only one type of arrow, which means that there is only one generator. This arrow is double-sided, which means that this generator is of order 2. If this is the Cayley diagram of a group, the group should have order 2, not 4.

(e) 

Solution: No. There is only one type of arrow, which means that there is only one generator. This arrow has order 4 because we see that four arrows form a 4-cycle. If this is the Cayley diagram of a group, the group should have order 4, not 8.

11. Answer the following questions about permutations and the symmetric group.

(a) Write as a product of disjoint cycles: $(1\ 5\ 2)\,(1\ 2\ 3\ 4)\,(1\ 3\ 5) =$

Solution: $(1)(2)(3\ 4)(5) = (3\ 4)$

(b) Write $(1\ 2\ 3\ 4)$ as a product of *transpositions* (i.e., 2-cycles).

Solution: $(12)(13)(14) = (23)(24)(12) = (34)(31)(23) = (14)(24)(34)$

(c) What is the *inverse* of the element $(1\ 3\ 2\ 6)\,(4\ 5)$ in S_6?

Solution: $(45)(1623)$

(d) The *order* of an element $g \in G$ is defined to be $|\langle g\rangle|$. Note that this (if finite) is also the minimum $k > 0$ such that $g^k = e$. What is the order of the element $(1\ 2\ 3\ 6)\,(4\ 5\ 7)$ in S_7?

Solution: The order is 12 because $[(1\ 2\ 3\ 6)\,(4\ 5\ 7)]^i \neq id$ for $i = 1, 2, \ldots, 11$ and $[(1\ 2\ 3\ 6)\,(4\ 5\ 7)]^{12} = id$

(e) Find an element of order 20 in S_9.

Solution: $(1\ 2\ 3\ 4\ 5)\,(6\ 7\ 8\ 9)$