1. (a) Let $n > 1$. Let A_n and B_n denote the set of even permutations and the set of odd permutations, respectively. Define a map $f : A_n \to B_n$ by $f(\pi) = (1\,2)\pi$ for all $\pi \in A_n$. We proved that f is surjective (see notes 2.3 Symmetric groups and alternating groups).
   Prove that this map is injective.
   (b) Let H be a subgroup of a group G, and let $x \in G$. Define a bijective map f from H to xH.
   (c) Show that this map is surjective.
   (d) Suppose G is a non-abelian group of order 1000 and H is a subgroup of order 20. Let x be an element of G which is not in H. (i) How many elements are in the left coset xH? (ii) How many elements are in the right coset Hx?
   How many left cosets of H are there?

2. a. Find all subgroups of D_4, and arrange them in a Hasse diagram, or subgroup lattice. Moreover, label each edge between $K \leq H$ with the index, $[H : K]$.
   b. Is $f\langle r\rangle = \langle r\rangle f$? What about other left and right cosets of $\langle r\rangle$? Prove your answer.
   c. Is the left coset $r^3 f\langle r^2, f\rangle$ equal to the right coset $\langle r^2, f\rangle r^3 f$?

3. For each statement below, determine if it is true or false. Prove your answer.
   (a) If the order of a group G is infinite (that is, if there are infinitely many elements in G), then the order of every $x \in G$ is also infinite. Recall that the order of x is the size of its orbit $\langle x\rangle$.
   (b) Every cyclic group is abelian.
   (c) Every abelian group is cyclic.
   (d) Every dihedral group is abelian.
   (e) Every dihedral group is not abelian.
   (f) There is a cyclic group of order 100.
   (g) There is a symmetric group of order 100
   (h) If some pair of non-identity elements in a group commute, then the group is abelian.
   (i) If every pair of elements in a group commute, the group is cyclic.
   (j) If every pair of elements in a group commute, the group is abelian.

4. (a) Is there a dihedral group of order 27?
   (b) If an alternating group A_n has order M, what order does the symmetric group S_n have?

5. For each part below, compute the orbit of the element in the group. Your answer should be a list of elements from the group that ends with the identity.
   (a) The element r^2 in the group D_{10}
   (b) The element 10 in the cyclic group $C_{16} = \mathbb{Z}/16$
   (c) The element 25 in the group $C_{30} = \mathbb{Z}/30$
   (d) The element 12 in the group $C_{42} = \mathbb{Z}/42$

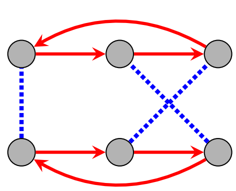
6. Open Group Explorer and sort the group library by order, with the smallest groups on top. Find a group which is not in any of the families of abelian/dihedral/symmetric/alternating groups, and explain why it's not in any of these families.

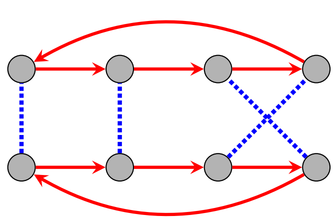
7. Recall that $\mathbb{Z}$ is a group under the operation of ordinary addition.
   (a) Create a Cayley diagram for it.
   (b) Is it abelian?
   (c) Give a minimal generating set consisting of more than one element.
   (d) Give an object whose symmetries form $(\mathbb{Z}, +)$.

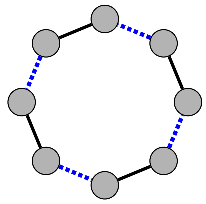
8. (a) Is there a group (of order larger than 1) in which no element (other than the identity) is its own inverse?
   (b) Find a group (of order larger than 1) such that there is only one solution to the equation $x^2 = e$, that is, the solution $x = e$, or explain why no such group exists.
   (c) Find a group that has exactly two solutions to the equation $x^2 = e$, or explain why no such group exists.
   (d) Find a group that has more than two solutions to the equation $x^2 = e$, or explain why no such group exists.
   (e) Find a group with at least two elements in it, and only one solution to the equation $x^3 = e$ (that is, the solution $x = e$) or explain why no such group exists.
   (f) Find a group that has more than two solutions to the equation $x^3 = e$, or explain why no such group exists.
   (g) There are two non-isomorphic groups of order 6. What are their names? Specify which, if any, are abelian.
   (h) Suppose m is a positive integer. If there exists only one group of order m, to what family must this group belong? Why?

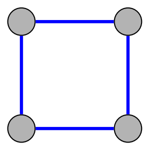
9. (a) If H is a subgroup of G and $a \in G$, then a left coset aH is ... [give the definition]
   (b) If G is a finite group, the *index* $[G : H]$ of a subgroup $H \leq G$ is [give a definition, not a theorem!] ...

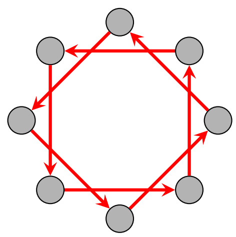
10. Determine whether each of the following diagrams are Cayley diagrams. If the answer is "yes," say what familiar group it represents, including the generating set. If the answer is "no," explain why.

(a) 

(b) 

(c) 

(d) 

(e) 

11. Answer the following questions about permutations and the symmetric group.

    (a) Write as a product of disjoint cycles: $(1\ 5\ 2)\,(1\ 2\ 3\ 4)\,(1\ 3\ 5) =$

    (b) Write $(1\ 2\ 3\ 4)$ as a product of *transpositions* (i.e., 2-cycles).

    (c) What is the *inverse* of the element $(1\ 3\ 2\ 6)\,(4\ 5)$ in S_6?

    (d) The *order* of an element $g \in G$ is defined to be $|\langle g \rangle|$. Note that this (if finite) is also the minimum $k > 0$ such that $g^k = e$. What is the order of the element $(1\ 2\ 3\ 6)\,(4\ 5\ 7)$ in S_7?

    (e) Find an element of order 20 in S_9.