Day 7 Abstract Algebra Notes

Ch 8 External direct products

New groups from old

Direct product of groups $(G, *)$ and $(H, \cdot)$ is a new group w/

set: $G \times H = \{ (g,h) : g \in G, h \in H \}$
Cartesian product

binary operation: $(g,h) * (g', h') = (g * g', h \cdot h')$

Identity: (e_G, e_H)

Inverse of (g,h) is (g^{-1}, h^{-1})

Ex Write the Cayley table for $\mathbb{Z}_2 \times \mathbb{Z}_3 = \{ (0,0), (0,1), (0,2), (1,0), (1,1), (1,2) \}$

	(0,0)	(0,1)	(0,2)	(1,0)	(1,1)	(1,2)
(0,0)	(0,0)	(0,1)	(0,2)	(1,0)	(1,1)	(1,2)
(0,1)	(0,1)	(0,2)	(0,0)	(1,1)	(1,2)	(1,0)
(0,2)	(0,2)	(0,0)	(0,1)	(1,2)	(1,0)	(1,1)
(1,0)	(1,0)	(1,1)	(1,2)	(0,0)	(0,1)	(0,2)
(1,1)	(1,1)	(1,2)	(1,0)	(0,1)	(0,2)	(0,0)
(1,2)	(1,2)	(1,0)	(1,1)	(0,2)	(0,0)	(0,1)

Cayley diagrams of direct products

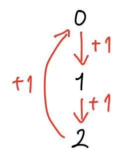
Let A, B be groups, and let e_A & e_B denote the identities of A & B (respectively).

Given a Cayley diagram of group A w/ generators $a_1, a_2, \dots, a_k$ and a Cayley diagram of group B w/ generators $b_1, b_2, \dots, b_\ell$, we can construct a Cayley diagram for direct product $A \times B$:

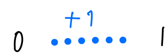
Vertices: (a,b) for each $a \in A$, $b \in B$
(often arranged in a rectangular grid)

Generators: $(a_1, e_B), \dots, (a_k, e_B)$ and $(e_A, b_1), \dots, (e_A, b_\ell)$.

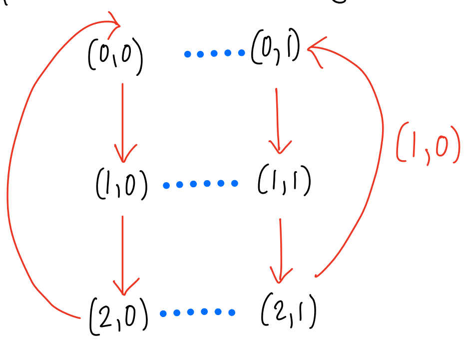
Ex: Cayley diagram for $\mathbb{Z}_3$ with generator 1:



Cayley diagram for $\mathbb{Z}_2$ with generator 1:



Cayley diagram for $\mathbb{Z}_3 \times \mathbb{Z}_2$ w/ generators $(0,1)$ and $(1,0)$:



Prop If $H \leq A$ and $K \leq B$ then $H \times K$ is a subgroup of $A \times B$.

Ex $\mathbb{Z}_3 \times \{0\}$ is a subgroup of $\mathbb{Z}_3 \times \mathbb{Z}_2$

$\{0, 2, 4\} \times \{0, 3\}$ is a subgroup of $\mathbb{Z}_6 \times \mathbb{Z}_6$

Exercise: Draw the Cayley diagram for

* $\mathbb{Z}_3 \times \mathbb{Z}_2$ using $\langle (1,1) \rangle$
* $\mathbb{Z}_4 \times \mathbb{Z}_2$ using $S = \{(1,0), (0,1)\}$
* Can you find an elt a s.t. $\mathbb{Z}_4 \times \mathbb{Z}_2 = \langle a \rangle$?

(pg 158 Thm 8.1 Order of an elt in a direct product)

Theorem If $G_1, G_2, \ldots, G_n$ are finite groups, then the order of $(g_1, g_2, \ldots, g_n) \in G_1 \times G_2 \times \cdots \times G_n$ is

$$\operatorname{lcm}(|g_1|, |g_2|, \ldots, |g_n|)$$

Proof (for $n=2$)

Let e_1, e_2 denote the identities of G_1, G_2 (respectively).

Let $s = \operatorname{lcm}(|g_1|, |g_2|)$, $t = |(g_1, g_2)|$.

Then $(g_1, g_2)^s = (\underbrace{g_1 \cdots g_1}_{s \text{ times}}, \underbrace{g_2 \cdots g_2}_{s \text{ times}})$ by def of external direct product

$= (e_1, e_2)$ since s is a multiple of $|g_1|$ and of $|g_2|$

This shows $\underbrace{|(g_1, g_2)|}_{t} \leq s$.

To show the other inequality, note that $(g_1^t, g_2^t) = (g_1, g_2)^t$ by def of direct product

$= (e_1, e_2)$ since $t = |(g_1, g_2)|$

So $g_1^t = e_1$ and $g_2^t = e_2$

So t is a multiple of $|g_1|$ and of $|g_2|$.

So $\underbrace{\operatorname{lcm}(|g_1|, |g_2|)}_{s} \leq t$ $\square$

(Ex 5 pg 158)

Ex How many elts of order 5 are there in $\mathbb{Z}_{25} \times \mathbb{Z}_5$?

Sol: By Thm 8.1, $(a,b) \in \mathbb{Z}_{25} \times \mathbb{Z}_5$ has order 5

iff $\operatorname{lcm}(|a|,|b|)$.

So <u>Case 1</u> $|a| = 5$ and $|b| = 1$ or 5,

OR <u>Case 2</u> $|a| = 1$ and $|b| = 5$.

<u>Case 1</u> $|a| = 5$: $5, 10, 15, 20$ (four possibilities)

$|b| = 1$ or 5 : all elts of $\mathbb{Z}_5$ (five possibilities)

<u>Case 2</u> $|a| = 1$: only the identity (one possibility)

$|b| = 5$: all elts of $\mathbb{Z}_5$ except for 0

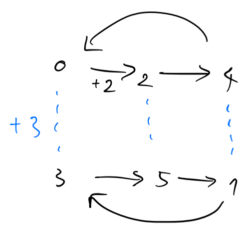
(four possibilities)

So $\mathbb{Z}_{25} \times \mathbb{Z}_5$ has $20 + 4 = 24$ elts of order 5

$$\boxed{\text{Finite \& finitely generated abelian groups}}$$

Note: $\mathbb{Z}_6 \cong \mathbb{Z}_3 \times \mathbb{Z}_2$ but $\mathbb{Z}_8 \not\cong \mathbb{Z}_2 \times \mathbb{Z}_4$

or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$



because $\mathbb{Z}_8$ has an elt of order 8, the number 1,

but every elt x in

$\mathbb{Z}_2 \times \mathbb{Z}_4$ & $(\mathbb{Z}_2)^3$

satisfies $x^4 = 0$.

(Thm 8.2 Criterion for $G \times H$ to be cyclic, pg 159)

$\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic iff m and n are relatively prime

(proof below)

Prove backward direction

(If $\gcd(n, m) = 1$ then $\mathbb{Z}_{nm} \cong \mathbb{Z}_n \times \mathbb{Z}_m$)

Suppose $\gcd(n, m) = 1$.

<u>Claim:</u> $(1,1) \in \mathbb{Z}_n \times \mathbb{Z}_m$ has order nm.

Let k be the order of $(1,1) \in \mathbb{Z}_n \times \mathbb{Z}_m$

Then $\underbrace{(1,1) + (1,1) + \cdots + (1,1)}_{k} = (k, k) = 0$

This means n divides k and m divides k.

So $k = \operatorname{lcm}(n, m)$.

But since $\gcd(n,m) = 1$, $\operatorname{lcm}(n,m) = nm$.

Since we know (from def of direct products) that the order of $\mathbb{Z}_n \times \mathbb{Z}_m$ is nm,

$\langle (1,1) \rangle$ must generate $\mathbb{Z}_n \times \mathbb{Z}_m$.

So $\mathbb{Z}_n \times \mathbb{Z}_m$ is a cyclic group of order nm, thus it is isomorphic to $\mathbb{Z}_{nm}$.

Prove forward direction

(If $\mathbb{Z}_{nm} \cong \mathbb{Z}_n \times \mathbb{Z}_m$ then $\gcd(n,m) = 1$)

<u>Pf</u> Suppose $\mathbb{Z}_{nm} \cong \mathbb{Z}_n \times \mathbb{Z}_m$.

$\left.\begin{array}{l}\text{Then } \mathbb{Z}_n \times \mathbb{Z}_m \text{ has an elt } (a,b) \text{ of order } nm\end{array}\right\} (*)$
(Since $1 \in \mathbb{Z}_{nm}$ has order nm).

For convenience, switch to "multiplicative notation".

Let C_n denote a cyclic group of order n, and let C_m denote a cyclic group of order m.

Let e_1 and e_2 denote the identities of C_n and C_m, resp.

Let $a \in C_n$ and $b \in C_m$ such that
$C_n = \langle a \rangle$ and $C_m = \langle b \rangle$

Then $a^n = e_1$ and if $0 < j < n$ then $a^j \neq e_1$

$b^m = e_2$ and if $0 < j < m$ then $b^j \neq e_2$

Then the order of (a,b) must be the smallest multiple of n and of m, $\operatorname{lcm}(n,m)$.

Since (a,b) has order nm (from $(*)$), $\operatorname{lcm}(n,m) = nm$. So the greatest common divisor of n and m is 1. $\square$

Ch 11 Fundamental Thm of Finite Abelian Groups

(Thm 11.1)

Every finite abelian group A is isomorphic to a direct product of cyclic groups. i.e.

$$A \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_j}$$

where each n_i is a prime power,

i.e. $n_i = p_i^{d_i}$ where p_i is prime, $d_i \in \mathbb{Z}_{>0}$

Ex Up to isomorphism, there are 6 abelian groups of order $200 = 2^3 \cdot 5^2$

$= 2 \cdot 2 \cdot 2 \cdot 5 \cdot 5$

By Prime powers		By divisors (Applying Thm 8.2)
$\mathbb{Z}_{200} \cong$ (by Thm 8.2) $\mathbb{Z}_8 \times \mathbb{Z}_{25}$	$222 \mid 55$	$\mathbb{Z}_{200}$
$\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_{25}$	$2 \mid 22 \mid 55$	$\mathbb{Z}_{100} \times \mathbb{Z}_2$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{25}$	$2 \mid 2 \mid 2 \mid 55$	$\mathbb{Z}_{50} \times \mathbb{Z}_2 \times \mathbb{Z}_2$
$\mathbb{Z}_{40} \times \mathbb{Z}_5 \cong \mathbb{Z}_8 \times \mathbb{Z}_5 \times \mathbb{Z}_5$	$222 \mid 5 \mid 5$	$\mathbb{Z}_{40} \times \mathbb{Z}_5$
$\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_5 \times \mathbb{Z}_5$	$2 \mid 22 \mid 5 \mid 5$	$\mathbb{Z}_{20} \times \mathbb{Z}_{10}$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5$	$2 \mid 2 \mid 2 \mid 5 \mid 5$	$\mathbb{Z}_{10} \times \mathbb{Z}_{10} \times \mathbb{Z}_2$

Classification of finitely generated abelian group

Every finitely generated abelian group A is isomorphic to a direct product of cyclic groups, i.e

$$A \cong \underbrace{\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}} \times \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_j}$$

Non abelian groups are much more mysterious

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