Motivation: Lagrange's Thm (order of a subgroup divides the order of the group)

# Ch 7 Cosets & Lagrange's Thm

## I. Properties of cosets

Warm-up: Given a subgroup $H$ of a group, we can define an equivalence relation $\sim_L$ on $G$ as follows:

$$x \sim_L y \quad \text{iff} \quad x^{-1}y \in H$$

($L$ stands for "left")

Ex: If $H = \{e\}$, then $x \sim_L y$ iff $x^{-1}y = e$ iff $x = y$,

So each equivalence class has exactly one elt.

If $H = G$, then $x \sim_L y$ for all $x, y \in G$,

so there is exactly one equivalence class, and it contains all elts of $G$.

In general, for a subgroup $H$, what are the equivalence classes corresponding to this equivalence relation $\sim_L$?

Answer: Given $g \in G$, the class $[g]_{\sim_L}$ containing $g$ is

$$[g]_{\sim_L} = \{x \in G : g \sim_L x\}$$
$$= \{x \in G : g^{-1}x \in H\}$$

$$= \{x \in G : g^{-1}x = h \text{ for some } h \in H\}$$

$$= \boxed{\{x \in G : x = gh \text{ for some } h \in H\}}$$

A natural way to denote this equivalence class is $gH$

Def Let $G$ be a group, $H$ a subgroup, $g \in G$.

The left coset of $H$ in $G$ containing $g$ (or with representative $g$) is

$$gH \overset{\text{def}}{=} \{gh : h \in H\}$$

Similarly, the right coset of $H$ containing $g$ is

$$Hg \overset{\text{def}}{=} \{hg : h \in H\}$$

Given a set $S$, let $|S|$ denote the number of elts in $S$

Note: If $G$ is abelian, $gH = Hg$.

Thm Let $G$ be a group, $H$ a subgroup. The left cosets of $H$ in $G$ partition $G$.

Pf The left cosets are equivalence classes for $\sim_L$, $gH = [g]_{\sim_L}$. $\square$

Note $eH = H$ is the only coset which is a group

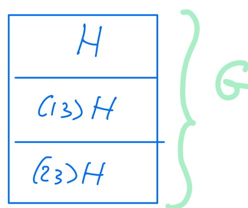
Ex $G = S_3$, $H = \langle (12) \rangle = \{Id, (12)\}$

Left cosets of $H$ in $G$:

① $H = \{e, (12)\}$ $= eH = (12)H$

② $(13)H = \{(13), \underset{(123)}{(13)(12)}\}$ $= (123)H$

③ $(23)H = \{(23), \underset{(132)}{(23)(12)}\}$ $= (132)H$



Right cosets of $H$ in $G$ (different!)

① $H = \{e, (12)\}$ $= eH = (12)H$

② $H(13) = \{(13), (132)\}$ $= H(132)$

③ $H(23) = \{(23), (123)\}$ $= H(123)$

| $G$ |
|---|
| $H$ |
| $H(13)$ |
| $H(23)$ |

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Ex $G = S_3$, $K = \langle (123) \rangle = \{e, (123), (132)\} = A_3 = \{\text{even permutations on } [n]\}$

Left cosets of $K$ in $G$:

① $K = \{e, (123), (132)\}$ $= (123)K = (132)K$

② $(12)K = \{(12), \underset{(12)(123)}{(23)}, \underset{(12)(132)}{(13)}\}$ $= (23)K = (13)K$

| | | | |
|---|---|---|---|
| $K$ | $e$ | $(123)$ | $(132)$ |
| $(12)K$ | $(12)$ | $(23)$ | $(13)$ |

Right cosets of $K$ in $G$ (the same!)

① $K = \{e, (123), (132)\}$ $= K(123) = K(132)$

② $K(12) = \{(12), (23), (13)\}$ $= K(23) = K(13)$

| | | | |
|---|---|---|---|
| $K$ | $e$ | $(123)$ | $(132)$ |
| $K(12)$ | $(12)$ | $(23)$ | $(13)$ |

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Ex $G = \mathbb{Z}$, $H = 4\mathbb{Z} = \{4k : k \in \mathbb{Z}\}$

Left (also right) cosets of $H$ in $G$:

① $4\mathbb{Z}$ $H$ $\ldots\ -8\ \ -4\ \ 0\ \ 4\ \ 8\ \ldots$

② $1 + 4\mathbb{Z}$ $1+H$ $\ldots\ -7\ \ -3\ \ 1\ \ 5\ \ 9\ \ldots$

③ $2 + 4\mathbb{Z}$ $2+H$ $\ldots\ -6\ \ -2\ \ 2\ \ 6\ \ 10\ \ldots$

④ $3 + 4\mathbb{Z}$ $3+H$ $\ldots\ -5\ \ -1\ \ 3\ \ 7\ \ 11\ \ldots$

All integers appear in my infinite array

## Lemma (Properties of cosets)

Let $G$ be a group, $H$ a subgroup, and $a, b \in G$.
Then the following conditions are equivalent.

1. $aH = bH$
2. $Ha^{-1} = Hb^{-1}$
3. $aH \subset bH$
4. $b \in aH$
5. $a^{-1}b \in H$

**Proof** We prove 1 implies 2

Suppose $aH = bH$

First we will show $Ha^{-1} \subset Hb^{-1}$.

Let $x \in Ha^{-1}$. (Goal: show $x \in Hb^{-1}$)

Since $aH = bH$, we have $a = ae = bh_b$ for some $h_b \in H$.

Then $x = h_a a^{-1}$ for some $h_a \in H$ (since $x \in Ha^{-1}$)

$$= h_a (h_b^{-1} b^{-1}) \quad (\text{since } a = bh_b)$$

$$= (h_a h_b^{-1}) b^{-1}$$

$$\in Hb^{-1} \quad (\text{since } h_a h_b^{-1} \in H)$$

So $Ha^{-1} \subset Hb^{-1}$.

Exercise: prove that $Hb^{-1} \subset Ha^{-1}$ to finish the proof that 1 implies 2

1 implies 3 by definition.

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We prove that 3 implies 1:

Suppose $aH \subset bH$. (We need to show $bH \subset aH$.)

Let $x \in bH$. (Goal: show $x \in aH$.)

Since $aH \subset bH$, we have $a = ae = bh_1$ for some $h_1 \in H$.

So $ah_1^{-1} = b$

Then $x = bh_2$ for some $h_2 \in H$ (since $x \in bH$)

$= ah_1^{-1}h_2$ since $b = ah_1^{-1}$

$= a(h_1^{-1}h_2)$

$\in aH$ (since $h_1^{-1}h_2 \in H$)

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We prove that 5 implies 4:

$a^{-1}b \in H$ $\quad$ $b \in aH$

Suppose $a^{-1}b \in H$.

Then $a^{-1}b = h$ for some $h \in H$

So $b = ah$, implying $b \in aH$.

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Exercise: prove the rest. $\square$

Ex $G = \mathbb{R}^3 = \left\{ \text{vectors } \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x, y, z \in \mathbb{R} \right\}$ Ex on pg 141

is an additive group under vector addition

Let $H$ be a plane through the origin

ex $H = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : 2x + 3y + 4z = 0 \right\}$

Then the coset $\begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} + H$ is $\left\{ \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} + \begin{bmatrix} x \\ y \\ z \end{bmatrix} : 2x + 3y + 4z = 0 \right\}$,

the plane that passes through $\begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$ and parallel to $H$.

So the cosets of $H$ partition $\mathbb{R}^3$ into planes parallel to $H$.

(In linear algebra, $H$ is called a subspace of $\mathbb{R}^3$ and the cosets $\begin{bmatrix} a \\ b \\ c \end{bmatrix} + H$ are called affine spaces)

## Lemma Properties of Cosets pg 139

*Let H be a subgroup of G, and let a and b belong to G. Then,*

1. $a \in aH$.
2. $aH = H$ if and only if $a \in H$.
3. $(ab)H = a(bH)$ and $H(ab) = (Ha)b$.
4. $aH = bH$ if and only if $a \in bH$.
5. $aH = bH$ or $aH \cap bH = \varnothing$.
6. $aH = bH$ if and only if $a^{-1}b \in H$.
7. $|aH| = |bH|$.
8. $aH = Ha$ if and only if $H = aHa^{-1}$.
9. $aH$ is a subgroup of $G$ if and only if $a \in H$.

## II. Lagrange's Theorem & consequences

**Def** Let $G$ be a group, $H$ a subgroup.
The index of $H$ in $G$, denoted $[G:H]$
is the number of left cosets of $H$ in $G$.

**Thm**
The number of left cosets of $H$ in $G$ is the same as
the number of right cosets of $H$ in $G$.

So $[G:H]$ is also the number of right cosets of $H$ in $G$.

**Proof**
Define a map $f: \{\text{left cosets}\} \longrightarrow \{\text{right cosets}\}$

by $\quad f: gH \longmapsto Hg^{-1}$

and we'll prove that it's a bijection.

* We need to check that this map is well-defined
(that is, we need to check that
$g_1H = g_2H$ implies $f(g_1H) = f(g_2H)$)
By Lemma , if $g_1H = g_2H$ then $Hg_1^{-1} = Hg_2^{-1}$ so $f(g_1H) = f(g_2H)$.

* To show that $f$ is injective, suppose $f(g_1H) = f(g_2H)$.
Then $Hg_1^{-1} = Hg_2^{-1}$. By Lemma, we have $g_1H = g_2H$.

* The map is surjective since $f(g^{-1}H) = Hg$ $\square$

**Prop** (Book pg 139 property 7)

Let $G$ be a group, $H$ a subgroup, $g \in G$. Then $|gH| = |H|$

**Proof**

Define a map $f: H \longrightarrow gH$

by $f: h \longmapsto gh$

and we'll prove that it's a bijection.

* To show that $f$ is injective, suppose $f(h_1) = f(h_2)$. Then $gh_1 = gh_2$. Multiplying by $g^{-1}$ on the left gives us $h_1 = h_2$.

* The map is surjective since every elt of $gH$ is of the form $gh$ for some $h \in H$, and $gh = f(h)$. $\square$

By a similar argument, prove $|Hg| = |H|$

**Thm** (Lagrange's Theorem) (Thm 7.1 pg 142)

Let $G$ be a finite group, $H$ a subgroup.

Then $\dfrac{|G|}{|H|} = [G:H]$.

In particular, $|H|$ divides $|G|$.

**Pf** The group $G$ is partitioned into $[G:H]$ distinct left cosets. Each left coset has $|H|$ elts. Hence $|G| = [G:H] \cdot H$ $\square$

**Cor** (Corollary 2 on pg 143)

Let $G$ be a finite, $g \in G$. Then $|g|$ divides $|G|$.

**Proof** Exercise

**Cor** (Corollary 3 on pg 143)

If $|G| = p$ is prime, then

① $G$ is cyclic

② any non-identity $g \in G$ is a generator.

(Every group of prime order is cyclic)

**Proof**

Since $p \geqslant 2$, there is some non-identity $g \in G$.

Then $|g|$ divides $p$ by above Cor 1.

Since $g \neq e$, $|g| \neq 1$, so $|g| = p$.

Therefore $\langle g \rangle$ has order $p$, so $\langle g \rangle = G$. $\square$

**Cor** If $K \leq H \leq G$

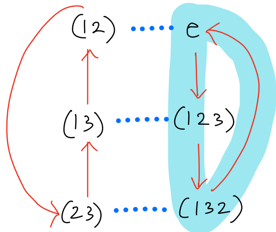
($K$ is a subgroup of $H$, and $H$ is a subgroup of $G$),

then $[G:K] = [G:H][H:K]$

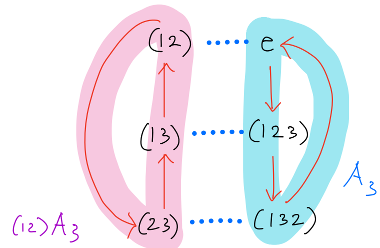
**Proof** $[G:K] = \dfrac{|G|}{|K|} = \dfrac{|G|}{|H|} \cdot \dfrac{|H|}{|K|} = [G:H][H:K]$ $\square$

## We can visualize subgroups and cosets

**Ex** the subgroup $A_3 = \langle (123) \rangle$ of $S_3$:



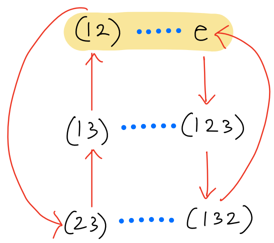

The two left cosets of $A_3$:



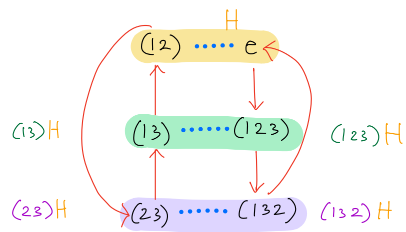

**Prop** If $\underbrace{[G:H]}_{\text{number of left cosets}} = 2$ then $gH = Hg$ for all $g \in G$.

**Prof** The left cosets $eH = H$ and $gH$ partition $G$, & the right cosets $He = H$ and $Hg$ partition $G$.
So $gH = Hg$.

the subgroup $\langle (12) \rangle$ of $S_3$:



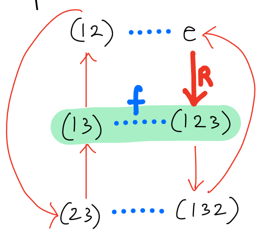

The three left cosets of $H$:



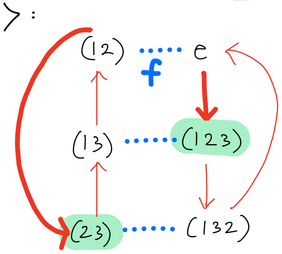

**Note** * To visualize the left coset $gH$ in a Cayley graph, start from vertex $g$ and follow all paths in $H$.

* Compare the left and right cosets $(123)H$ vs $H(123)$:

– Left coset $R\langle f \rangle$: the vertices the $f$ arrows can reach after the path to $R$ has been followed




– Right coset $\langle f \rangle R$: the vertices the $R$ arrows can reach from elts in $\langle f \rangle$:




– The left cosets of $H$ look like copies of $H$

– The right cosets are usually scattered

only because we adopted the convention that arrows mean right multiplication

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