Abstract Algebra Notes Day 5 Tue, Oct 7 2025

Ch 6 Isomorphisms part II

Starting 7:50pm

Thm 6.2 (Properties of isomorphisms acting on elts)

Let $\varphi : G \longrightarrow H$ be a isomomorphism of groups.

① φ Sends e_G to e_H

② For each $x \in G$, $\varphi(x^k) = [\varphi(x)]^k$, in particular:

For each $x \in G$, $\varphi(x^{-1}) = [\varphi(x)]^{-1}$

③ $G = \langle a \rangle$ if and only if $H = \langle \varphi(a) \rangle$

④ $|a| = |\varphi(a)|$ for all $a \in G$

(isomorphisms preserve orders)

Proof ① See Day 4 notes

②
$$\begin{aligned} e_H &= \varphi(e_G) \\ &= \varphi(x\, x^{-1}) \\ &= \varphi(x)\, \varphi(x^{-1}) \end{aligned}$$

Multiply both sides on the left by $[\varphi(x)]^{-1}$:

$$[\varphi(x)]^{-1} e_H = [\varphi(x)]^{-1} \varphi(x)\ \varphi(x^{-1})$$

$$[\varphi(x)]^{-1} = \varphi(x^{-1})$$

③ ($\Rightarrow$) Suppose $G = \langle a \rangle$.

Then $[\varphi(a)]^k \in H$ by closure property of group H,

so $\langle \varphi(a) \rangle \subset H$.

Because φ is onto, for any elt $b \in H$,

there is an elt $x \in G$ such that $\varphi(x) = b$.

Note $x = a^k$ for some k since $G = \langle a \rangle$.

Thus $b = \varphi(x) = \varphi(a^k) \underset{\text{part (2)}}{=} [\varphi(a)]^k$

So $b \in \langle \varphi(a) \rangle$ for every $b \in H$,

so $H = \langle \varphi(a) \rangle$.

($\Leftarrow$) Suppose $H = \langle \varphi(a) \rangle$.

$\langle a \rangle \subseteq G$ by def.

To show $G \subseteq \langle a \rangle$, let $x \in G$.

Then $\varphi(x) \in H = \langle \varphi(a) \rangle$

since H is the codomain

So we have $\varphi(x) = (\varphi(a))^k$ for some integer k

by def of $\langle \varphi(a) \rangle$.

So $\varphi(x) = (\varphi(a))^k = \varphi(a^k)$

Since φ is injective, $x = a^k$

Hence $x \in \langle a \rangle$.

So $G = \langle a \rangle$. $\square$

④ Follows from the fact that $\varphi(a^n) = [\varphi(a)]^n$

and $\varphi(e_G) = e_H$. $\square$

Thm 6.3 (Properties of isomorphisms acting on groups)

Let $\varphi: G \to H$ be an isomorphism.

1. The inverse map $\varphi^{-1}: H \to G$ is an isomorphism
2. G is abelian iff H is abelian
3. G is cyclic iff H is cyclic
4. If K is a subgroup of G then the image $\varphi(K)$ is a subgroup of H.
5. If J is a subgroup of H then the preimage $\varphi^{-1}(J)$ is a subgroup of G.

Ex · $\mathbb{Z}_{12}$ isn't isomorphic to D_6 or A_4 since $\mathbb{Z}_{12}$ is abelian but the others aren't

· D_6 has an elt of order 6: the rotation by $\frac{2\pi}{6}$

· A_4 consists of even permutations in S_4: (1), 3-cycles, and elts of the form $(ab)(cd)$, which have order 1, 3, and 2 (respectively)

So D_6 and A_4 are not isomorphic.

Prop Let G, H, K be groups.

① $\mathrm{id}_G : G \to G$ is an isomorphism
$\qquad x \mapsto x$

[Pf: id_G a homomorphism: $\mathrm{id}_G(ab) = ab = \mathrm{id}_G(a)\,\mathrm{id}_G(b)$. ✓
id_G a bijection. ✓]

② If $\varphi : G \to H$ is an isomorphism, then the inverse bijection φ^{-1} is also an isomorphism.

[Pf The inverse bijection φ^{-1} is a bijection. ✓

Given $c, d \in H$, $c = \varphi(a)$ and $d = \varphi(b)$ for some $a, b \in G$ since φ is a bijection.

Then $\varphi(ab) = \varphi(a)\,\varphi(b)$ since φ is a homomorphism
$\qquad\qquad = c \quad d$

So $\varphi^{-1}(cd) = ab = \varphi^{-1}(c)\,\varphi^{-1}(d)$, and thus φ^{-1} is a homomorphism. ✓ □]

③ (The composition of two isomorphisms is also an isomorphism)

If $\overset{\text{phi}}{\varphi} : G \to H$ and $\overset{\text{psi}}{\psi} : H \to K$ are isomorphisms, then $\psi \circ \varphi : G \to K$ is an isomorphism.

[Proof Composition of bijections is a bijection. ✓

For $a, b \in G$, $\psi(\varphi(ab)) = \psi(\varphi(a)\varphi(b))$ since φ is a homomorphism
$\qquad = \psi(\varphi(a))\,\psi(\varphi(b))$ since ψ is a homomorphism. ✓]

Remark The set $\mathrm{Aut}(G) \overset{\text{def}}{=} \{\text{automorphisms of } G\}$ forms a group under composition. It's called the <u>automorphism group of G</u>.

Def Let G be a group, and let $a \in G$.

The function $\phi_a : G \to G$

defined by $x \mapsto axa^{-1}$

is called <u>the inner automorphism of G induced by a</u>.

Exercise: prove that each ϕ is an automorphism of G.
(HW 04)

Rem Recall from HW 2 that aHa^{-1} is a subgroup

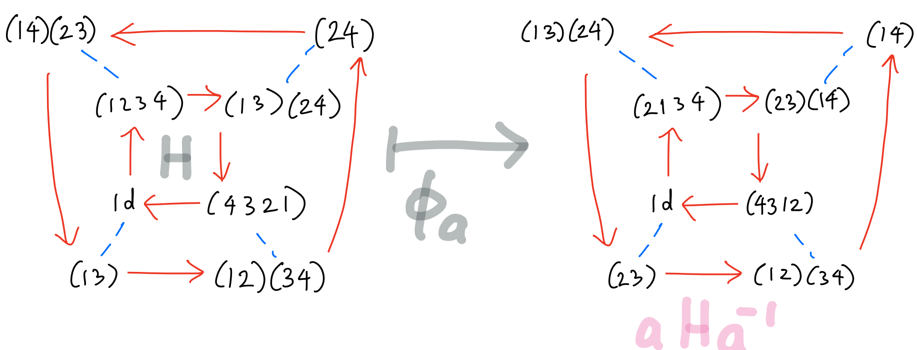
for any $a \in G$ and subgroup H of G.

$$\phi_a : H \longrightarrow aHa^{-1}$$
$$h \longmapsto aha^{-1}$$

gives an isomorphism from H onto aHa^{-1}

Ex Consider $G = S_{100}$

$$H = \langle (1234), (13) \rangle = \{ \text{Id}, (1234), \overbrace{(13)(24)}^{(1234)^2}, (4321), (13), \underbrace{(12)(34)}_{(13)(1234)}, (24), (14)(23) \}$$



let $a = (12)$

$$aHa^{-1} = \langle (12)(1234)(12), \ (12)(13)(12) \rangle = \langle (2134), (23) \rangle$$

Fact: $\sigma (1234\ldots k) \sigma^{-1} = (\sigma(1) \ \sigma(2) \ldots \sigma(k))$

Proof See HW 05

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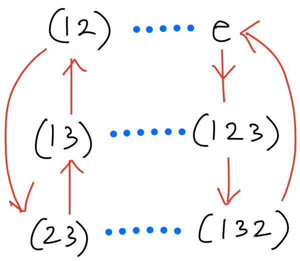
~~HW 05~~

Consider $G = S_{100}$

Let $H = \langle (123), (12) \rangle$.

(i) Draw the Cayley diag



Hint: The Cayley diagram for this is given at the beginning of Day 4 notes

Apply the isomorphism

$\phi_a : H \longrightarrow aHa^{-1}$ for $a = (456)$

Draw Cayley diagram aHa^{-1}

(see ex from class)

Ans same

(ii) Apply the isomorphism

$\phi_a : H \rightarrow aHa^{-1}$ for $a = (1\,4)$