Groups acting on sets & Thm 6.1 Cayley's Theorem

Intuitively, a group G "acts" on a set S of configurations by "naturally permuting" the configurations in S.

Ex 1 (Extra Ex)

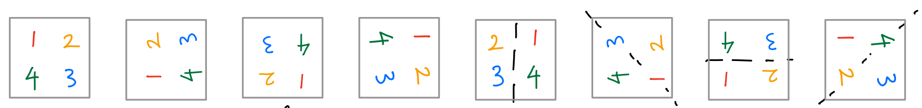
$G = D_4$ = Symmetry group of a square

$= \{ e, \; R, \; R^2, \; R^3, \; f, \; fR, \; fR^2, \; fR^3 \} = \langle R, f \rangle$

R = counterclockwise rotation by $90°$,

f = horizontal flip $\longleftrightarrow$

Configurations of the square mattress (corners labeled by 1, 2, 3, 4):

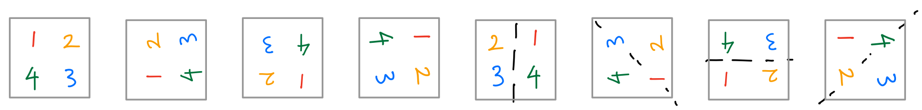


Init $\quad R(\text{Init}) \quad R^2(\text{Init}) \quad R^3(\text{Init}) \quad f(\text{Init}) \quad fR(\text{Init}) \quad fR^2(\text{Init}) \quad fR^3(\text{Init})$

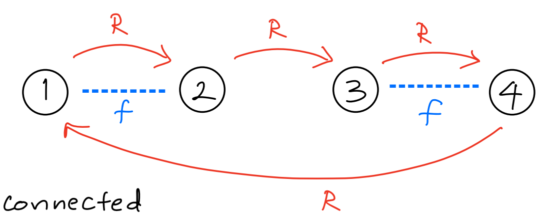
Let $X = \{1, 2, 3, 4\}$, just a set.

We can think of D_4 as the following permutations in S_4:

e	R	R^2	R^3	f	fR	fR^2	fR^3
e	$(1\,2\,3\,4)$	$(1\,3)(2\,4)$	$(1\,4\,3\,2)$	$(1\,2)(3\,4)$	$(1\,3)$	$(1\,4)(2\,3)$	$(2\,4)$



Action diagram:



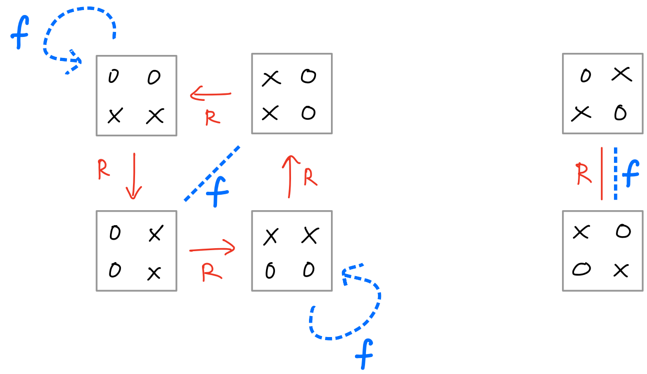
Note: This diagram is connected

Ex 2 $G = D_4 = \langle R, f \rangle$ again

$\underline{X} = \{$ squares w/ two corners labeled $\times$ and $\}$
$\qquad\qquad\qquad\qquad\qquad$ " $\qquad\qquad\qquad\qquad\qquad$ o

$$= \left\{ \boxed{\begin{matrix} o & o \\ \times & \times \end{matrix}}\ \boxed{\begin{matrix} o & \times \\ o & \times \end{matrix}}\ \boxed{\begin{matrix} \times & \times \\ o & o \end{matrix}}\ \boxed{\begin{matrix} \times & o \\ \times & o \end{matrix}}\ \boxed{\begin{matrix} o & \times \\ \times & o \end{matrix}}\ \boxed{\begin{matrix} \times & o \\ o & \times \end{matrix}} \right\}$$

Action diagram :



Note: This diagram is $\underline{\underline{\text{not}}}$ connected

Remark (Extra): When the action diagram is connected, the group action is called <u>transitive</u> (from one elt a in the set $\underline{X}$, we can get to all other elts in $\underline{X}$ using G)

Def Let G be a group and let $\underline{X}$ be a set.

A <u>left action</u> of G on $\underline{X}$ is a map

$$G \times \underline{X} \longrightarrow \underline{X}$$

$$(g, x) \longmapsto \underbrace{gx}_{\text{symbol}}$$

where

1. $ex = x$ for all $x \in \underline{X}$
2. $(ba)x = b(ax)$ for all $x \in \underline{X}$ and $a, b \in G$.

A set $\underline{X}$ equipped w/ such a map is called a left <u>G-set</u>.

(Extra info)

Note: X doesn't need to be related to G in any way. But group actions are more interesting when the G-sets X is related to the group G in some way.

Ex 3 (Ex of group actions from linear algebra) (Extra Ex)

$G = GL_2(\mathbb{R}) = \{2\times 2 \text{ invertible matrices w/ real entries}\}$

$X = \mathbb{R}^2 = \{\text{vectors } \begin{bmatrix} x \\ y \end{bmatrix} : x, y \in \mathbb{R}\}$

Then G acts on X by left multiplication

$$GL_2(\mathbb{R}) \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix} \right) \longmapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Verify that this map satisfies the two conditions:

1. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ for all $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$

2. For all $A, B \in GL_2(\mathbb{R})$, we have

$$(BA)\begin{bmatrix} x \\ y \end{bmatrix} = B\left(A\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

since matrix multiplication is associative.

Ex 1 and 2 are group actions for D_4

Def $\operatorname{Perm}(X) =$ the group of permutations of X

bijections $X \to X$

(This is the def used in class)

Note: If $|X| = n$ then $\operatorname{Perm}(X) \cong S_n$

An alternative definition of group action

Intuition

- Given a group G, we have a "switchboard" with a button $\boxed{g}$ for every $g \in G$.
- Given $a \in G$, pressing the button $\boxed{a}$ rearranges the objects in our set X. This gives a permutation on X; call this $\overset{\text{phi}}{\phi}(a) \in \operatorname{Perm}(X)$
- Given $b \in G$, pressing the button $\boxed{b}$ also rearranges the objects in our set X. Call this permutation $\phi(b)$.
- The element $ba \in G$ also has a button $\boxed{ba}$.
- For G to act on X, we require that pressing the $\boxed{ba}$ button gives the same rearrangement of X as first pressing the $\boxed{a}$ button, followed by the button $\boxed{b}$, that is,

$$\phi(ba) = \phi(b)\,\phi(a) \qquad (*)$$

for all $a, b \in G$

Alternative def of group action

A left action of G on X is a group homomorphism

$$\phi : G \longrightarrow \operatorname{Perm}(X)$$

Note: A right action requires $\phi(ba) = \phi(a)\phi(b)$ instead of $(*)$

Back to Ex 1

"Switchboard" for D_4:

$\boxed{e}$	$\boxed{R}$	$\boxed{R^2}$	$\boxed{R^3}$
$\boxed{f}$	$\boxed{fR}$	$\boxed{fR^2}$	$\boxed{fR^3}$

Pressing the $\boxed{R}$ button permutes $X = \{1, 2, 3, 4\}$ as follows:

$\phi(R)$:

$$\begin{tikzcd} 1 \arrow[r, bend left, "R"] & 2 \arrow[r, bend left, "R"] & 3 \arrow[r, bend left, "R"] & 4 \arrow[lll, bend left, "R"] \end{tikzcd}$$

$\phi(R) = (1\,2\,3\,4)$

Pressing the $\boxed{f}$ button permutes $X = \{1, 2, 3, 4\}$ as follows:

$\phi(f)$:

$$\begin{tikzcd} 1 \arrow[r, bend left, dashed, "f"] & 2 \arrow[l, bend left, dashed, "f"] & 3 \arrow[r, bend left, dashed, "f"] & 4 \arrow[l, bend left, dashed, "f"] \end{tikzcd}$$

$\phi(f) = (1\,2)(3\,4)$

$$\phi(D_4) = \{e, (1\,2\,3\,4), (1\,3)(2\,4), (1\,4\,3\,2), (1\,2)(3\,4), (1\,3), (1\,4)(2\,3), (2\,4)\} \underset{\text{subgroup}}{\leq} S_4$$

> Exercise: In Example 2, think of the xxoo squares as $1, 2, \dots, 6$.
> Let $\phi: D_4 \to S_6$ be the left action of D_4 on $\{1, 2, \dots, 6\}$.
> Write the elements of $\operatorname{im}\phi$. What familiar group is $\operatorname{im}\phi$ isomorphic to?

Next, we see that every group acts on itself by left multiplication.

<u>Def</u> Let $\mathbb{X} = G$. Define the homomorphism

$$\phi : G \longrightarrow \overbrace{\text{Perm}(G)}^{\text{Perm}(\mathbb{X})} \quad \text{by}$$
$$g \longmapsto T_g$$

where

$$T_g : G \longrightarrow G$$
$$x \longmapsto gx \text{ for all } x \in G$$

(that is, T_g is multiplication by g on the left)

In other words, pressing the $\boxed{g}$ button on our "group switchboard" multiplies every elt on the left by g.

The image $\phi(G) = \{T_g : g \in G\}$ is called <u>the left regular representation</u> of G

Exercise: Prove that ϕ is a group homomorphism

Prove that ϕ is injective

Recall: The units in $\mathbb{Z}_{12}$ are $1, 5, 7, 11$, & they form a group w/ operation $\cdot$

<u>Ex</u> Calculate the left regular rep of $U(12) = \{1, 5, 7, 11\}$

T_1 = multp by 1 = $\begin{bmatrix} 1 & 5 & 7 & 11 \\ 1 & 5 & 7 & 11 \end{bmatrix}$

T_5 = multp by 5 = $\begin{bmatrix} 1 & 5 & 7 & 11 \\ 5 & 1 & 11 & 7 \end{bmatrix}$ $= (ab)(cd)$

T_7 = multp by 7 = $\begin{bmatrix} 1 & 5 & 7 & 11 \\ 7 & 11 & 1 & 5 \end{bmatrix}$

$(ac)(bd)$

T_{11} = $\begin{bmatrix} 1 & 5 & 7 & 11 \\ 11 & 7 & 5 & 1 \end{bmatrix}$

$(ad)(bc)$

<u>Note</u> Every group also acts on itself by right multiplication. The corresponding action diagram is equal to the Cayley diagram.

<u>Cayley's Thm</u> (See Thm 6.1) Every group is isomorphic to a group of permutations.

<u>Pf</u> Let ϕ be the left regular representation

$$\phi : G \longrightarrow \mathrm{Perm}(G)$$
$$g \longmapsto T_g$$

given above.

Then $\operatorname{im}\phi$ is group of permutations.

Since ϕ is injective, we have $G \cong \operatorname{im}\phi$. $\square$

<u>Two contrasting reasons why Cayley's Thm is important</u>

① Cayley's Thm allows us to represent an abstract group in a concrete way

② Present-day set of axioms for a group is the correct abstraction of a group of permutations.

<u>Note:</u> • Concepts similar to "group acting on set" appear in other algebraic structures.

• For example, in ring theory, we have "R-modules" instead of "G-sets".

ring group

— end —