Ref: Book Ch 14 Part II: Quotient rings; Part III: Prime ideals and maximal ideals

Lemma 1. Let H be an additive subgroup of the group $(G, +)$. Then the following are equivalent:

(1) $a + H = b + H$ (2) $b \in a + H$ (3) $b - a \in H$

1 Multiplication of cosets is well-defined

(1) What is a coset (for rings)? (Answer: Read and take notes on the last paragraph on pg 250)

(2) Let R be a ring, let $s, s', t, t' \in R$, and let I be an ideal of R. Suppose also that

$$s' \in s + I \text{ and } t' \in t + I.$$

Homework: Prove that

$$s't' \in st + I \tag{1}$$

(You can follow the first paragraph of the proof of Theorem 14.2 on pg 251.)

Solution: Since $s' \in s + I$, we have $s' = s + a$ for some $a \in I$. Similarly, since $t' \in t + I$, we have $t' = t + b$ for some $b \in I$. Then

$$s't' = (s + a)(t + b) = st + (sb + at + ab)$$

We have $sb, at, ab \in I$ because I is an ideal (and therefore satisfies the "absorbing" property). Therefore $sb + at + ab \in I$ since I is a subring (and therefore is closed under the ring multiplication operation for R). So $s't' \in st + I$.

Note that (1) implies $s't' + I = st + I$, due to Lemma 1.

2 Quotient rings (aka factor rings)

(1) Write the full statement of the reverse direction of Theorem 14.2 ("If A is an ideal of R then the set of cosets $r + A$ is a ring under the given additive and multiplicative operations of cosets)

(2) What is the zero element in a quotient ring R/I?

Solution: The zero element in the quotient ring R/I is the identity element of the additive quotient group $(R/I, +)$. The identity element in the quotient group $(R/I, +)$ is the subgroup I. So the zero element in the quotient ring R/I is the ideal I, thought of as a coset in R/I.

(3) Take notes on Examples 8 and 9 on pg 251.

3 Example 10

Consider the set

$$\text{Mat}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\}$$

of 2×2 matrices with integer entries, under the usual matrix addition and matrix multiplication. Let I be the subset consisting of matrices with even entries. You showed in past HW that I is an ideal of $\text{Mat}_2(\mathbb{Z})$.

(1) Consider the cosets

$$\begin{pmatrix} 7 & 8 \\ 5 & -3 \end{pmatrix} + I \text{ and } \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + I.$$

Are they the same element in the quotient ring $\text{Mat}_2(\mathbb{Z})/I$? Explain.

Solution: Yes. Why? (See explanation Example 10)

(2) Are the cosets $\begin{pmatrix} 2 & 8 \\ 5 & -3 \end{pmatrix} + I$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + I$ the same set? Explain.

Solution: No. Why? (See explanation Example 10)

(3) What are the elements (the cosets) in the quotient ring $\text{Mat}_2(\mathbb{Z})/I$? How many are there?

Solution: We have $\text{Mat}_2(\mathbb{Z})/I = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} + I : a, b, c, d \in \{0, 1\} \right\}$. It consists of $16 = 2^4$ cosets.

(4) What is the zero element in the quotient ring $\text{Mat}_2(\mathbb{Z})/I$? Describe this set of matrices.

Solution: Earlier, we said that the zero element in the quotient ring R/I is the ideal I. So the zero element in this case is I, the set of 2×2 matrices with even entries.

(5) What's the unity element of the quotient ring $\text{Mat}_2(\mathbb{Z})/I$? Describe the elements in this coset.

Solution: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + I = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, d \text{ are odd integers}, b, c \text{ are even integers}\} \right\}$

4 Example 12

Take notes of Example 12 on pg 252-253 (quotient ring of $\mathbb{R}[x]$ which is "the same" as the ring $\mathbb{C}$)

5 Prime and maximal ideals

(1) What is the definition of prime ideal and maximal ideal? (pg 253)

(2) Read Example 13 on pg 254, and use this info to write three ideals of $\mathbb{Z}$ which are prime ideals, write three ideals of $\mathbb{Z}$ which are not prime ideals.

(3) Write down (and understand) the first two sentences of Example 15 (about the ideal $\langle x^2+1\rangle$ of $\mathbb{R}[x]$).

(4) Write and understand the statements of Theorem 14.3 ("determining whether an ideal is prime") and 14.4 ("determining whether an ideal is maximal") on pg 255.

(5) Prove that the quotient ring $\mathbb{Z}[x]/\langle x\rangle$ is an integral domain.

> **Solution:** First, prove that the ideal $\langle x\rangle$ is prime (You can follow the proof given in Example 17). By Theorem 14.3, $\mathbb{Z}[x]/\langle x\rangle$ is an integral domain.

6 Required for Math 5210 students only:

(1) Write the proof of the statement of Example 15.

(2) Write sketch of proof of Theorem 14.3

(3) Write sketch of proof of Theorem 14.4