1 Cancellation law in an integral domain

Suppose R is an integral domain and $x \in R$. If $x^2 = x$, what are the possible values of x?

Solution: Suppose $x^2 = x$. Then $xx = x1$. If x is nonzero, then the cancellation law of integral domain tells us that $x = 1$. Conclusion: $\boxed{x \text{ can either be the zero element or the unity element}}$.

2 First, write down the definition of the *characteristic* of a ring.

(1) Suppose R is a ring with unity $\mathbf{1}$. (a) Prove the following: If $\mathbf{1}$ has order n under addition, then the characteristic of R is n.

(b) If $\mathbf{1}$ is of infinite order under addition, what is the characteristic of R?

Solution: You can follow the proof in Gallian Theorem 13.3 on page 243.

(2) What is the characteristic of the ring $\mathbb{R}$ of real numbers?

Solution: The order of the unity element 1 is infinite, so the characteristic of the ring is 0.

(3) What is the characteristic of the ring $\mathbb{Z}_6$?

Solution: The order of the unity element $\mathbf{1}$ is 6, so the characteristic of $\mathbb{Z}_6$ is 6.

3 Matrices with integer entries

Definition 1. Consider the set

$$\mathrm{Mat}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\}$$

of 2×2 matrices with integer entries. It forms a ring with unity $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ under the usual matrix addition and matrix multiplication. The zero element is the zero matrix $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

Let I be the subset of $\mathrm{Mat}_2(\mathbb{Z})$ consisting of matrices with even entries. Prove that

$$I \text{ is an ideal of } \mathrm{Mat}_2(\mathbb{Z}).$$

(You need to show that:

- I is an additive subgroup of $\mathrm{Mat}_2(\mathbb{Z})$
- I "absorbs" all elements of $\mathrm{Mat}_2(\mathbb{Z})$, that is, for all $a \in I$ and $r \in \mathrm{Mat}_2(\mathbb{Z})$, we have $ar \in I$ and $ra \in I$.)

4 Principal ideal definition

If R is a commutative wring with unity, write what it means for a subset I of R to be a principal ideal of R.

(Note: If you can show that a subset I can be written this way, you do not need to check the two conditions in the "Ideal test" above.)

5 An ideal of the ring of integers

Consider the subset

$$n\mathbb{Z} = \{nk : k \in \mathbb{Z}\} = \{..., -2n, -n, 0, n, 2n, ...\}$$

of the ring $\mathbb{Z}$ of integers.

Is $n\mathbb{Z}$ an ideal? Is $n\mathbb{Z}$ a principal ideal? If it is, describe an element of $n\mathbb{Z}$ which generates $n\mathbb{Z}$

Solution: $n\mathbb{Z} = \langle n \rangle$ is the principal ideal generated by n.

6 An ideal of the ring of integer polynomials?

Let $\mathbb{Z}[x]$ denote the ring of all polynomials having integer coefficients.

(1) Consider the subset S of $\mathbb{Z}[x]$ of integer polynomials $f(x)$ such that $f(5) = 0$, that is, integer polynomials which have 5 as a root.

What do the polynomials in S look like? Give some examples.

Is S an ideal? If S is a principal ideal, describe an element of S which generates S.

Solution: Yes. $S = \langle x - 5 \rangle$

(2) Consider the subset T of $\mathbb{Z}[x]$ of polynomials $f(x)$ such that $f(0) = 5$.

What do the polynomials in T look like? Give some examples.

Is T an ideal? (Explain)

Solution: No, it's not even a subring. For example, $f(x) = x + 5$ is in T, but the product $f(x)f(x) = x^2 + 10x + 25$ is not in T.

7 Ideals?

Which of the following subsets of $\mathbb{Z}[x]$ are ideals? Answer **Yes** or **No**.

- If you answer No, provide a specific example of how the subset fails the absorbing property of an ideal or how the subset fails to be an additive subgroup of $\mathbb{Z}[x]$.
- If you answer Yes, explain why the absorbing property holds (you don't need to prove that the subset is an additive group).

(1) $S = \mathbb{Z}$, that is, all the constant polynomials in $\mathbb{Z}[x]$.

Solution: No, S is not an ideal of $\mathbb{Z}[x]$ because it fails the absorbing property. For example, $f(x) = 5$ is a polynomial in S and $g(x) = x^2$ is a polynomial in $\mathbb{Z}[x]$, but their product is $5x^2$ which is not in S.

(2) S is the set consisting of the constant zero function and of all polynomials with no constant term.

Solution: Yes, S can be written in set-builder notation as $\{f(x)x : x \in \mathbb{Z}[x]\}$, which shows that S is the principal ideal generated by the polynomial x. Notation: $\langle x \rangle$.

Note that we don't need to do the "ideal test" because we see that S is a principal ideal (and therefore an ideal).

(3) The set S of integer polynomials $f(x)$ such that $f'(2) = 0$, i.e. 2 is a root of $f'(x)$.

Solution: No, S is not an ideal because it fails the "absorbing" property. For example, $f(x) = x^2 - 4x$ is a polynomial in S (since $f'(x) = 2x - 4$), and $g(x) = x$ is a polynomial in $\mathbb{Z}[x]$, however, their product is $x^3 - 4x^2$ which has derivative $3x^2 - 4x$ which is not in S.

(4) The set S of integer polynomials $f(x)$ such that $f(r) \geq 0$ for all real number r (when you graph the polynomial, the curve is always on or above the x-axis).

Solution: No. The subset S is not an ideal because it fails the absorbing property. For example, $f(x) = x^2$ is a polynomial in S and $g(x) = -3$ is a polynomial in $\mathbb{Z}[x]$, but their product is $-3x^2$ which is not in S.

(5) The set S of integer polynomials $f(x)$ such that $f(1) \neq 0$, i.e. 1 is *not* a root of $f(x)$.

Solution: No. The subset S is not an ideal because it fails the absorbing property. For example, $f(x) = x^2$ is a polynomial in S and $g(x) = (x - 1)$ is a polynomial in $\mathbb{Z}[x]$, but their product is $(x - 1)x^2$ which is not in S.

(6) The set S of integer polynomials $f(x)$ whose coefficients are all even integers.

Solution: Yes, the absorbing property holds because any integer polynomial multiplied by a polynomial in S has even integer coefficients.

8 An ideal of the ring of real polynomials

Consider the ring $\mathbb{R}[x]$ of polynomials with real coefficients, and let I denote the set of polynomials in $\mathbb{R}[x]$ with no constant term and no term of degree 1. For example,

$$p(x) = \pi x^2 - ex^5 \in I,$$

but

$$q(x) = \pi - ex^5 \text{ and } r(x) = \pi x + x^2 \text{ are not in } I.$$

Is I an ideal of $\mathbb{R}[x]$? Is I a principal ideal of $\mathbb{R}[x]$? If it is, describe an element of I which generates I.

Solution: $I = \langle x^2 \rangle$ is the principal ideal generated by x^2.