1 First isomorphism theorem (example)

Recall that $\mathbb{C}^*$ denotes the set of nonzero complex numbers equipped with usual multiplication as the binary operation. Consider the group homomorphism $f : \mathbb{C}^* \to \mathbb{R}^*$ defined by

$$f : z \mapsto |z|$$

Since $|a + bi| = \sqrt{a^2 + b^2}$, this is the same as saying that $f(a + bi) = \sqrt{a^2 + b^2}$ for all $a + bi \in \mathbb{C}^*$. You can use the fact that $|xy| = |x||y|$ for all complex numbers x, y.

(a) What is $\ker f$? Is f injective?

> **Solution:** $\{z \in \mathbb{C}^* : |z| = 1\}$, that is, the circle group $\mathbb{T}$.

(b) Pick a coset of $\ker(f)$ not equal to $\ker(f)$, for example, $2\ker(\phi)$ or $5i\ker(\phi)$. Write this coset as the fiber of an element in $\mathrm{Im} f$.

> **Solution:** Let K denote the kernel of f. For example, the coset $3iK$ is equal to $\{3iz : |z| = 1\}$. So every element $3iz$ in this coset is sent to $f(3iz) = |3||i||z| = 3 \cdot 1 \cdot 1 = 3$. So $\boxed{3iK = f^{-1}(\{3\})}$.

(c) What is $\mathrm{Im} f$?

> **Solution:** $\mathbb{R}^*_{>0}$, the subgroup of $\mathbb{R}^*$ consisting of all positive real numbers.

(d) What does the first isomorphism theorem tell us?

> **Solution:** The quotient group $\mathbb{C}^*/\mathbb{T}$ is isomorphic to $\mathbb{R}^*_{>0}$.

2 Required for Math 5210 students only

Do Ch 10 Problems 41 & 42 (2nd & 3rd isomorphism theorems). You can use the proofs in Judson Sec 11.2

3 Definitions

Write down (and memorize) the definition of ...
zero element, unity (or identity), ring, commutative ring, ring with unity (or ring with identity), integral domain, field, zero divisor, and unit.

4 Gaussian integers

(1) Write down the definition of the set $\mathbb{Z}[i]$ of the Gaussian integers.

(2) Is $\mathbb{Z}[i]$ a subring of the ring of complex numbers (under usual addition and multiplication)?

> **Solution:** Yes. All four properties of being a subring are satisfied.

(3) Is $\mathbb{Z}[i]$ a commutative ring? Is $\mathbb{Z}[i]$ an integral domain?

> **Solution:** See Examples at the beginning of Chapter 13

(4) What are the units of $\mathbb{Z}[i]$?

> **Solution:** See class notes

(5) Is $\mathbb{Z}[i]$ a field?

> **Solution:** No because not all nonzero elements are units. (A field is a commutative ring with unity such that every nonzero elements are units.)

5 Question

Definition 1. Consider the set

$$\mathrm{Mat}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\}$$

of 2×2 matrices with integer entries. It forms a (non-commutative) ring with unity under the usual matrix addition and matrix multiplication. The unity of $\mathrm{Mat}_2(\mathbb{Z})$ is the identity matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and the zero element is the zero matrix $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

For each of the following subsets S of $\mathrm{Mat}_2(\mathbb{Z})$, answer whether S is a subring of $\mathrm{Mat}_2(\mathbb{Z})$. (Answer **Yes/ No**)

If you claim S is not a subring, specify which subring conditions are not satisfied (S doesn't contain the zero element; S is not closed under ring addition; S is not closed under ring negation; S is not closed under ring multiplication)

(1) S is the subset of $\mathrm{Mat}_2(\mathbb{Z})$ consisting of invertible matrices.

> **Solution:** No, S is not a subring. It is closed under ring multiplication, but it fails the other three properties: S doesn't contain the zero element; S is not closed under ring addition; S is not closed under ring negation.

(2) S is the subset of $\mathrm{Mat}_2(\mathbb{Z})$ consisting of matrices with even entries.

(3) $S = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} : a, c, d \in \mathbb{Z} \right\}$ is the subset of lower-triangular matrices in $\mathrm{Mat}_2(\mathbb{Z})$.

(4) $S = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbb{Z} \right\}$ is the subset of diagonal matrices in $\mathrm{Mat}_2(\mathbb{Z})$.

Solution: For the last three sets, S is a subring. Prove that all four properties are satisfied.

6 Question

Definition 2. Let R be a ring which has unity denoted by the symbol $\mathbf{1}$. An element $u \in R$ is called a *unit* (also called an *invertible element*) if there exists $v \in R$ such that $uv = vu = \mathbf{1}$.

(1) Suppose R is a ring with unity $\mathbf{1}$. Prove the following: if $x^4 = \mathbf{0}$ then $\mathbf{1} - x$ is a unit.

Solution: Let $v = \mathbf{1} + x + x^2 + x^3$. Then we have

$$\begin{aligned}(\mathbf{1} - x)v &= (\mathbf{1} - x)(\mathbf{1} + x + x^2 + x^3) \\ &= \mathbf{1} + x + x^2 + x^3 - x(\mathbf{1} + x + x^2 + x^3) \\ &= \mathbf{1} - x^4 \\ &= \mathbf{1} \text{ since } x^4 = \mathbf{0}\end{aligned}$$

(2) What are the units (if any) in the ring $\mathbb{Z}_{10}$?

Solution: The units are the nonzero elements which are relatively prime to 10: $1, 3, 7, 9$. A notation that we have used all semester for this set of units is $U(10)$.

What are the zero divisors (if any) of the ring $\mathbb{Z}_{10}$?

Solution: All the nonzero elements which are not units, $\boxed{2,4,5,6,8}$, are zero divisors. For example, 4 is a zero divisor because $(4)(5) = 0$.

(3) Give an example of an infinite field. Give an example of a finite field.

Solution: Examples of infinite fields: $\mathbb{Q}, \mathbb{R}, \mathbb{C}$. For examples of a finite field, see page 239 (Chapter 13) about fields.