1 First isomorphism theorem (example)

Recall that $\mathbb{C}^*$ denotes the set of nonzero complex numbers equipped with usual multiplication as the binary operation. Consider the group homomorphism $f : \mathbb{C}^* \to \mathbb{R}^*$ defined by

$$f : z \mapsto |z|$$

Since $|a+bi| = \sqrt{a^2+b^2}$, this is the same as saying that $f(a+bi) = \sqrt{a^2+b^2}$ for all $a+bi \in \mathbb{C}^*$. You can use the fact that $|xy| = |x||y|$ for all complex numbers x, y.

(a) What is $\ker f$? Is f injective?

(b) Pick a coset of $\ker(f)$ not equal to $\ker(f)$, for example, $2\ker(\phi)$ or $5i\ker(\phi)$. Write this coset as the fiber of an element in $\mathrm{Im} f$.

(c) What is $\mathrm{Im} f$?

(d) What does the first isomorphism theorem tell us?

2 Required for Math 5210 students only

Do Ch 10 Problems 41 & 42 (2nd & 3rd isomorphism theorems). You can use the proofs in Judson Sec 11.2

3 Definitions

Write down (and memorize) the definition of ...
zero element, unity (or identity), ring, commutative ring, ring with unity (or ring with identity), integral domain, field, zero divisor, and unit.

4 Gaussian integers

(1) Write down the definition of the set $\mathbb{Z}[i]$ of the Gaussian integers.

(2) Is $\mathbb{Z}[i]$ a subring of the ring of complex numbers (under usual addition and multiplication)?

(3) Is $\mathbb{Z}[i]$ a commutative ring? Is $\mathbb{Z}[i]$ an integral domain?

(4) What are the units of $\mathbb{Z}[i]$?

(5) Is $\mathbb{Z}[i]$ a field?

5 Question

Definition 1. Consider the set

$$\mathrm{Mat}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\}$$

of 2×2 matrices with integer entries. It forms a (non-commutative) ring with unity under the usual matrix addition and matrix multiplication. The unity of $\mathrm{Mat}_2(\mathbb{Z})$ is the identity matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and the zero element is the zero matrix $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

For each of the following subsets S of $\mathrm{Mat}_2(\mathbb{Z})$, answer whether S is a subring of $\mathrm{Mat}_2(\mathbb{Z})$. (Answer **Yes/ No**)

If you claim S is not a subring, specify which subring conditions are not satisfied (S doesn't contain the zero element; S is not closed under ring addition; S is not closed under ring negation; S is not closed under ring multiplication)

(1) S is the subset of $\mathrm{Mat}_2(\mathbb{Z})$ consisting of invertible matrices.

(2) S is the subset of $\mathrm{Mat}_2(\mathbb{Z})$ consisting of matrices with even entries.

(3) $S = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} : a, c, d \in \mathbb{Z} \right\}$ is the subset of lower-triangular matrices in $\mathrm{Mat}_2(\mathbb{Z})$.

(4) $S = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbb{Z} \right\}$ is the subset of diagonal matrices in $\mathrm{Mat}_2(\mathbb{Z})$.

6 Question

Definition 2. Let R be a ring which has unity denoted by the symbol $\mathbf{1}$. An element $u \in R$ is called a *unit* (also called an *invertible element*) if there exists $v \in R$ such that $uv = vu = \mathbf{1}$.

(1) Suppose R is a ring with unity $\mathbf{1}$. Prove the following: if $x^4 = \mathbf{0}$ then $\mathbf{1} - x$ is a unit.

(2) What are the units (if any) in the ring $\mathbb{Z}_{10}$?

What are the zero divisors (if any) of the ring $\mathbb{Z}_{10}$?

(3) Give an example of an infinite field. Give an example of a finite field.