Math 4210/5210 Algebra HW 09. Submit hand-written solutions at the beginning of class. Partial solutions are given in a separate PDF file.

1 Reading.

(1) Given a homomorphism $f : G \to H$, how can you construct a normal subgroup of G using f?

> **Solution:** See the corollary on pg 198 (Chapter 10), or see class notes

(2) Read the blog post The First Isomorphism Theorem, Intuitively by Tai-Danae Bradley (Math3ma). Summarize or write some highlights based on the reading. Write about half a page.

2 Question.

Consider the homomorphism $f : \mathbb{R}^* \to \mathbb{R}^*$ defined by $f : x \mapsto x^2$.

(a) What is $\ker f$? Is f injective? If not, state whether it is a 2-to-1, or 3-to-1, or 4-to-1 mapping, etc.

> **Solution:** $\ker f = \{1, -1\}$

(b) Pick a coset of $\ker(f)$ not equal to $\ker(f)$, for example, $2\ker(\phi)$ or $(-3)\ker(\phi)$. Write down all elements of this coset, and then demonstrate that f sends all elements of this coset to the same element in the image of f. Write this coset as the fiber of an element in $\mathrm{Im} f$.

> **Solution:** Let $K = \ker f$. The coset $(-3)K$ is equal to $\{-3, 3\}$. Both elements in this coset are sent to 9 by f. So we can write the coset $(-3)K$ as the fiber $f^{-1}(\{9\})$.

(c) What is $\mathrm{Im} f$?

> **Solution:** $\mathbb{R}^*_{>0}$, the subgroup of $\mathbb{R}^*$ consisting of all positive real numbers.

3 Question.

Consider the homomorphism $f : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ defined by $f(x) = 3x$.

> **Solution:** See Chapter 10 Example 9 on pg 199.

(1) What is $\ker f$? For which t is f a t-to-one mapping? Is f injective?

> **Solution:** f is a 3-to-one mapping, since $\ker f = \{0, 4, 8\}$ is of order 3. It's not injective.

(2) Find all cosets of $\ker f$.

> **Solution:** Since $\mathbb{Z}_{12}$ has order 12 and $\ker f$ has order 3, there should be four cosets. Let K denote $\ker f$. Two of the cosets are K itself and $2 + K = \{2, 6, 10\}$. Find the other two cosets.

(3) Consider the coset $2 + \ker f$. Find all elements of this coset.

> **Solution:** The coset $2 + \ker f$ is equal to $\{2, 6, 10\}$

(4) Find all elements in the fiber $f^{-1}(\{6\})$. Is it true or false that the fiber $f^{-1}(\{6\})$ is a coset of $\ker f$?

> **Solution:** True. $f^{-1}(\{6\} = \{2, 6, 10\}$, which is the coset $2 + \ker f$.
>
> Note: In fact, Theorem 10.1 (property 6) on pg 196 (also class notes) tells us that every fiber $f^{-1}(\{y\})$ is a coset of $\ker f$, for every $y \in \text{Im} f$.

(5) What familiar group is the quotient group $\mathbb{Z}_{12}/\ker f$ isomorphic to?

> **Solution:** The quotient group $\mathbb{Z}_{12}/\{0, 4, 8\}$ is a cyclic group of order 4, since it can be generated by the coset $2 + \ker f$. So it's isomorphic to $\mathbb{Z}_4$.

(6) What is $\text{Im} f$?

> **Solution:** $\text{Im} f$=$\{3, 6, 9, 0\}$

4 Wrapping function.

Consider the wrapping function $f : \mathbb{R} \to \mathbb{C}^*$ defined by $f(\theta) = e^{i\theta}$.

(1) Compute $\ker f$.

(2) What is $f(\mathbb{R})$?

> **Solution:** The answers are given in Chapter 10 Example 15 on pg 202 or class notes.

5 Wrapping function from $\mathbb{Z}$.

Consider the integer version of the wrapping function $f : (\mathbb{Z}, +) \to \mathbb{C}^*$ defined by

$$f(m) = e^{i\frac{m\pi}{2}}$$

Since $i = e^{i\frac{\pi}{2}}$, we can write $e^{i\frac{m\pi}{2}} = (e^{i\frac{pi}{2}})^m = i^m$, so f can also be defined by

$$f(m) = i^m$$

(1) Find $\ker f$.

(2) Let $K = \ker f$. List all cosets of K.

(3) What familiar group is $\mathbb{Z}/K$ isomorphic to?

(4) Find $f(\mathbb{Z})$.

Solution: (1) The are infinitely many elements in $\ker f$.
(2) Hint: There are four cosets of K. List them all.
(3) Hint: If G is cyclic, then every quotient group of G is also cyclic (see Chapter 9 Exercise 11, pg 188)
(4) The image is $\mathbb{T}_4$, where

$$\mathbb{T}_4 \text{ is the set of all 4-th roots of unity } \{1, i, -1, -i\} = \{1, e^{i(\pi/2)}, e^{i(\pi/2)2}, e^{i(\pi/2)3}\}.$$

See Question 1 of the assignment “Algebra Typesetting Project 04: Isomorphisms”. See also discussion of complex numbers in Day 4 class notes.

Hint: The homomorphisms in Questions 6, 7 have the same structure, so your answers for both should be “the same”.

6 Question.

Let D_4 be the dihedral group of order 8

$$\begin{aligned} D_4 = \{&e, R, R^2, R^3, \\ &f, fR, fR^2, fR^3\} \end{aligned}$$

(1) Draw the Cayley graph of D_4 using f and R as generators.

(2) Let $V_4 = \{e, h, v, r\}$ be the (non-square) rectangle mattress group, where e is the identity element. Let $\phi : D_4 \to V_4$ be the homomorphism determined by

$$\phi(R) = h \text{ and } \phi(f) = v.$$

Using the homomorphism property $\phi(ab) = \phi(a)\phi(b)$, find where ϕ sends all elements of D_4.

Solution:

$$\phi(R^3) = \phi(R)\phi(R)\phi(R) = hhh = eh = h,$$
$$\phi(fR) = \phi(f)\phi(R) = vh = r.$$

I have given you where ϕ sends four of the elements. Compute where ϕ sends the other four elements.

(3) Find $\ker(\phi)$. Circle the elements of $\ker(\phi)$ in your Cayley graph.
Is ϕ injective? If not, state whether it is a 2-to-1 or 4-to-1 or 8-to-1 mapping.

Solution: No, $\ker(\phi) = \{e, R^2\}$ has cardinality 2, so ϕ is a two-to-one mapping.

(4) Pick a coset of $\ker(\phi)$ not equal to $\ker(\phi)$, for example, $R\ker(\phi)$ or $f\ker(\phi)$. Write down all elements of this coset, and then demonstrate that ϕ sends all elements of this coset to the same element in the codomain V_4. Write this coset as the fiber of an element in the image of ϕ.

Solution: For example, the coset $R\ker(\phi)$ is equal to $\{R, R^3\}$. Both elements in this coset are sent to h by ϕ. We can write the coset $R\ker(\phi)$ as the fiber $\phi^{-1}(\{h\})$.

(5) What familiar group is the quotient group $D_4/\ker\phi$ isomorphic to? Explain briefly (in one or two sentences).

(6) Find $\mathrm{Im}(\phi)$.

7 Required for Math 5210; only recommended for Math 4210 students.

Let $S_{[\pm 2]}$ be the set of all permutations on $\{-2, -1, 1, 2\}$. Consider the group

$$S_2^B = \{w \text{ in } S_{[\pm 2]} \text{ such that } w(-a) = -w(a)\}$$

under function composition.

(1) We write the elements of S_2^B in cycle notation.

- Let $\mathbf{f} = (\mathbf{1},\ \mathbf{2})(-\mathbf{1},\ -\mathbf{2})$
- Let $\mathbf{r} = (\mathbf{1},\ -\mathbf{2},\ -\mathbf{1},\ \mathbf{2})$.

Then we compute

- $\mathbf{fr} = (\mathbf{1},\ -\mathbf{1})$
- $\mathbf{fr^2} = (\mathbf{1},\ -\mathbf{2})(-\mathbf{1},\ \mathbf{2})$

Problem: Draw the Cayley diagram of S_2^B using $\mathbf{f}$ and $\mathbf{r}$ as generators. Label all vertices in cycle notation.

Solution: Hints: f, fr, fr^2 all have order 2, and r has order 4, and

$$S_2^B = \{e, r, r^2, r^3, \\ f, fr, fr^2, fr^3\}$$

(2) Consider the homomorphism $\varphi : S_2^B \to S_2^B$ where

$$\varphi(\mathbf{r}) = \mathbf{r}^2 \text{ and } \varphi(\mathbf{f}) = \mathbf{f}.$$

Using the homomorphism property $\varphi(ab) = \varphi(a)\varphi(b)$, find where φ sends all other elements of S_2^B.

Solution:

- $\varphi(\mathbf{r}^3) = \varphi(\mathbf{r})\varphi(\mathbf{r})\varphi(\mathbf{r}) = (\mathbf{r}^2)(\mathbf{r}^2)(\mathbf{r}^2) = e\mathbf{r}^2 = \mathbf{r}^2$, and
- $\varphi(\mathbf{fr}) = \varphi(\mathbf{f})\varphi(\mathbf{r}) = \mathbf{fr}^2 = \mathbf{r}$.

Compute where ϕ sends the other four elements.

(3) Find $\ker(\varphi)$. Circle the elements of $\ker(\varphi)$ in your Cayley graph.

Is φ injective? If not, state whether it is a 2-to-1 or 4-to-1 or 8-to-1 mapping.

(4) Let $K = \ker(\varphi)$. Pick a coset of K not equal to K, for example, $\mathbf{r}K$ or $\mathbf{f}K$ or $\mathbf{fr}K$. Write down all elements of this coset, and then demonstrate that φ sends all elements of this coset to the same element in the codomain S_2^B. Write this coset as the fiber of an element in the image of φ.

Solution: For example, the coset $\mathbf{r}\ker(\phi)$ is equal to $\{R, R^3\}$. Both elements in this coset are sent to r^2 by φ. We can write the coset $r\ker(\varphi)$ as the fiber $\varphi^{-1}(\{\mathbf{r}^2\})$.

Now, practice doing the same computation for a different coset of K. Try $\mathbf{f}K$ or $\mathbf{fr}K$.

(5) What familiar group is the quotient group $S_2^B/\ker\varphi$ isomorphic to? Explain briefly (in one or two sentences).

(6) Find $\mathrm{Im}(\varphi)$.