Math 4210/5210 Algebra HW 09. Submit hand-written solutions at the beginning of class.
Partial solutions are given in a separate PDF file.

1 Reading.

(1) Given a homomorphism $f : G \to H$, how can you construct a normal subgroup of G using f?

(2) Read the blog post The First Isomorphism Theorem, Intuitively by Tai-Danae Bradley (Math3ma). Summarize or write some highlights based on the reading. Write about half a page.

2 Question.

Consider the homomorphism $f : \mathbb{R}^* \to \mathbb{R}^*$ defined by $f : x \mapsto x^2$.

(a) What is $\ker f$? Is f injective? If not, state whether it is a 2-to-1, or 3-to-1, or 4-to-1 mapping, etc.

(b) Pick a coset of $\ker(f)$ not equal to $\ker(f)$, for example, $2\ker(\phi)$ or $(-3)\ker(\phi)$. Write down all elements of this coset, and then demonstrate that f sends all elements of this coset to the same element in the image of f. Write this coset as the fiber of an element in $\mathrm{Im} f$.

(c) What is $\mathrm{Im} f$?

3 Question.

Consider the homomorphism $f : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ defined by $f(x) = 3x$.

(1) What is $\ker f$? For which t is f a t-to-one mapping? Is f injective?

(2) Find all cosets of $\ker f$.

(3) Consider the coset $2 + \ker f$. Find all elements of this coset.

(4) Find all elements in the fiber $f^{-1}(\{6\})$. Is it true or false that the fiber $f^{-1}(\{6\})$ is a coset of $\ker f$?

(5) What familiar group is the quotient group $\mathbb{Z}_{12}/\ker f$ isomorphic to?

(6) What is $\mathrm{Im} f$?

4 Wrapping function.

Consider the wrapping function $f : \mathbb{R} \to \mathbb{C}^*$ defined by $f(\theta) = e^{i\theta}$.

(1) Compute $\ker f$.

(2) What is $f(\mathbb{R})$?

5 Wrapping function from $\mathbb{Z}$.

Consider the integer version of the wrapping function $f : (\mathbb{Z}, +) \to \mathbb{C}^*$ defined by

$$f(m) = e^{i\frac{m\pi}{2}}$$

Since $i = e^{i\frac{\pi}{2}}$, we can write $e^{i\frac{m\pi}{2}} = (e^{i\frac{pi}{2}})^m = i^m$, so f can also be defined by

$$f(m) = i^m$$

(1) Find $\ker f$.

(2) Let $K = \ker f$. List all cosets of K.

(3) What familiar group is $\mathbb{Z}/K$ isomorphic to?

(4) Find $f(\mathbb{Z})$.

Hint: The homomorphisms in Questions 6, 7 have the same structure, so your answers for both should be "the same".

6 Question.

Let D_4 be the dihedral group of order 8

$$\begin{aligned} D_4 = &\{e, R, R^2, R^3, \\ &f, fR, fR^2, fR^3\} \end{aligned}$$

(1) Draw the Cayley graph of D_4 using f and R as generators.

(2) Let $V_4 = \{e, h, v, r\}$ be the (non-square) rectangle mattress group, where e is the identity element. Let $\phi : D_4 \to V_4$ be the homomorphism determined by

$$\phi(R) = h \text{ and } \phi(f) = v.$$

Using the homomorphism property $\phi(ab) = \phi(a)\phi(b)$, find where ϕ sends all elements of D_4.

(3) Find $\ker(\phi)$. Circle the elements of $\ker(\phi)$ in your Cayley graph.

Is ϕ injective? If not, state whether it is a 2-to-1 or 4-to-1 or 8-to-1 mapping.

(4) Pick a coset of $\ker(\phi)$ not equal to $\ker(\phi)$, for example, $R\ker(\phi)$ or $f\ker(\phi)$. Write down all elements of this coset, and then demonstrate that ϕ sends all elements of this coset to the same element in the codomain V_4. Write this coset as the fiber of an element in the image of ϕ.

(5) What familiar group is the quotient group $D_4/\ker\phi$ isomorphic to? Explain briefly (in one or two sentences).

(6) Find $\mathrm{Im}(\phi)$.

7 Required for Math 5210; only recommended for Math 4210 students.

Let $S_{[\pm 2]}$ be the set of all permutations on $\{-2, -1, 1, 2\}$. Consider the group

$$S_2^B = \{w \text{ in } S_{[\pm 2]} \text{ such that } w(-a) = -w(a)\}$$

under function composition.

(1) We write the elements of S_2^B in cycle notation.

- Let $\mathbf{f} = (\mathbf{1},\ \mathbf{2})(-\mathbf{1},\ -\mathbf{2})$
- Let $\mathbf{r} = (\mathbf{1},\ -\mathbf{2},\ -\mathbf{1},\ \mathbf{2})$.

Then we compute

- $\mathbf{fr} = (\mathbf{1},\ -\mathbf{1})$
- $\mathbf{fr^2} = (\mathbf{1},\ -\mathbf{2})(-\mathbf{1},\ \mathbf{2})$

Problem: Draw the Cayley diagram of S_2^B using $\mathbf{f}$ and $\mathbf{r}$ as generators. Label all vertices in cycle notation.

(2) Consider the homomorphism $\varphi : S_2^B \to S_2^B$ where

$$\varphi(\mathbf{r}) = \mathbf{r}^2 \text{ and } \varphi(\mathbf{f}) = \mathbf{f}.$$

Using the homomorphism property $\varphi(ab) = \varphi(a)\varphi(b)$, find where φ sends all other elements of S_2^B.

(3) Find $\ker(\varphi)$. Circle the elements of $\ker(\varphi)$ in your Cayley graph.

Is φ injective? If not, state whether it is a 2-to-1 or 4-to-1 or 8-to-1 mapping.

(4) Let $K = \ker(\varphi)$. Pick a coset of K not equal to K, for example, $\mathbf{r}K$ or $\mathbf{f}K$ or $\mathbf{fr}K$. Write down all elements of this coset, and then demonstrate that φ sends all elements of this coset to the same element in the codomain S_2^B. Write this coset as the fiber of an element in the image of φ.

(5) What familiar group is the quotient group $S_2^B/\ker\varphi$ isomorphic to? Explain briefly (in one or two sentences).

(6) Find $\mathrm{Im}(\varphi)$.