

Math 4210/5210 Algebra HW 08. Submit hand-written solutions at the beginning of class, before Exam 2.

1 Properties of cosets

Let H be a subgroup of G . Recall that, if $x \in G$, the set $xH := \{xh \mid h \in H\}$ is a *left coset* of H .

- (i) Prove that if $b \in aH$, then $aH \supset bH$.
- (ii) Prove that if $b \in aH$, then $aH \subset bH$.
- (iii) Prove that if $c \in H$, then $cH \subset H$.
- (iv) Prove that if $c \in H$, then $cH \supset H$.

2 Lagrange's Theorem

- (i) Gallian Chapter 7 Exercise 9 (pg 150).
- (ii) The symmetric group S_8 is a subgroup of the symmetric group S_9 . How many left cosets of the subgroup S_8 are there in S_9 ? Give a brief (one or two sentences) explanation.
What is the index $[S_9 : S_8]$? (The answer should be a positive integer)

3 Direct products

(There are partial solutions to some of Gallian textbook exercises, on the back of the book.)

- (i) Gallian Chapter 8 Exercise 15 (pg 168)
- (ii) Gallian Chapter 8 Exercise 35 (pg 169)

4 Normal subgroups

Definition 4.1. Let G be a group. Fix elements $a, b \in G$ (possibly $a = b$). Let H be a subgroup of G . Let $aHbH$ denote the set

$$\{a h_1 b h_2 \mid h_1, h_2 \in H\} \quad (4.1)$$

Let abH denote the left coset $\{a b h \mid h \in H\}$. Note that both sets are always defined.

- (i) Demonstrate that in general the set $aHbH$ and the set abH are not the same. That is, give a specific group G , elements $a, b \in G$, and a subgroup H of G , then show that $aHbH \neq abH$.
(Hint: Use the example given in Chapter 9 Exercise 40 on pg 190).
- (ii) Give an example where “multiplication” of cosets is not well-defined. That is, give a group G and a subgroup H , then find cosets $a_1H = a_2H$ and $b_1H = b_2H$ where $a_1b_1H \neq a_2b_2H$.
- (iii) Consider the symmetric group S_3 and let $J := \langle (1\ 2) \rangle$. Then there are 3 left cosets of J :

$$\begin{aligned} J &= \{e, (1\ 2)\}, \\ (132)J &= (13)J = \{(1\ 3), (1\ 3\ 2)\}, \text{ and} \\ (1\ 2\ 3)J &= (2\ 3)J = \{(2\ 3), (1\ 2\ 3)\}. \end{aligned}$$

Compute all elements of the set $(1\ 3)J(1\ 3)J$ and the set $(1\ 3\ 2)J(1\ 3)J$.

This page is required for Math 5210 students. It's recommended (but not required) for Math 4210 students.

5 Question (Required for Math 5210 students)

For $n \geq 1$, let $[\pm n]$ denote the set $\{-n, -(n-1), \dots, -1\} \cup \{1, 2, \dots, n\}$. Let $S_{[\pm n]}$ denote the set of all bijections from $[\pm n]$ to $[\pm n]$. Define the set

$$S_n^B := \{\text{bijections } w : [\pm n] \rightarrow [\pm n] \text{ where } w(-a) = -w(a)\},$$

which is a subset of $S_{[\pm n]}$.

For example, let p be the function which swaps 2 and -4 such that $p(j) = j$ for all other numbers j . Then $p \in S_{[\pm 4]}$, but $p \notin S_4^B$ because $p(-2) = -2 \neq -4 = -p(2)$.

- (i) Give an example of a non-identity function which is in the set S_4^B .
- (ii) It is possible to verify that S_n^B is a group with function composition as the binary operation. People often use cycle notation to denote the functions in S_n^B . For example, consider S_2^B .

- The function $\mathbf{f} := (1, 2)(-1, -2)$ swaps 1 and 2; and swaps -1 and -2 .
- The function $\mathbf{g} := (1, -1)$ swaps 1 and -1 , and it fixes both 2 and -2 .
- $\mathbf{fg} = (1, 2)(-1, -2)(1, -1) = (\mathbf{1}, -\mathbf{2}, -\mathbf{1}, \mathbf{2})$ is a 4-cycle $1 \mapsto -2 \mapsto -1 \mapsto 2 \mapsto 1$.
- The function $(1 - 2)$ which swaps 1 and -2 but fixes all other numbers is *not* in S_n^B , but the function $\mathbf{gfg} = (\mathbf{1}, -\mathbf{2})(-\mathbf{1}, \mathbf{2})$ *is* in S_n^B .

Compute the following functions:

$$\mathbf{gfgf} = \quad \mathbf{gfgfg} = \quad \mathbf{gfgfgf} =$$

- (iii) Draw a Cayley diagram of S_2^B , and label all vertices in cycle notation. Use the functions f, g defined above as generators for your Cayley diagram.
Hint: S_2^B happens to be isomorphic to D_4 .
- (iv) Compute the center of S_2^B . (Hint: Use your knowledge about the center of D_4 .)
Let Z denote the center of S_2^B . What familiar group is S_2^B/Z isomorphic to?
- (v) If $n \geq 1$, give a formula for the order of S_n^B . (Hint: If $n \geq 3$, the group S_n^B is *not* a Dihedral group.)

Solutions to Question 1

- Proof for part (i) and (ii):

Suppose $b \in aH$. This means that $b = ac$ and $bc^{-1} = a$ for some specific $c \in H$. Note that $c^{-1} \in H$.

First, we show $aH \subset bH$: Let $h \in H$. Then $c^{-1}h \in H$ since H is a group. Then $ah = bc^{-1}h \in bH$ by definition.

Next, we show $aH \supset bH$: Let $h \in H$. Then $ch \in H$ since H is a group. Then $bh = ach \in aH$ by definition.

- Proof for part (iii) and (iv): See proof of Lemma Property (2) on pg. 139.