Math 4210/5210 Algebra HW 07. Submit hand-written solutions at the beginning of class.

Partial solutions to Questions 1, 3, and 4 are given on the next page.

1. Cayley diagrams of direct products

Draw the Cayley diagram for each of the following.

(1) $\mathbb{Z}_4 \times \mathbb{Z}_2$ with generators $(1, 0)$ (solid) and $(0, 1)$ (dotted),
(2) $\mathbb{Z}_4 \times \{0\}$ with generator $(1, 0)$,
(3) $\mathbb{Z}_2 \times \mathbb{Z}_2$ with generators $(1, 0)$ (solid) and $(0, 1)$ (dotted)
(4) $\mathbb{Z}_2 \times \mathbb{Z}$ with generators $(1, 0)$ (dotted) and $(0, 1)$ (solid)
(5) Draw the Cayley diagram for $\mathbb{Z}_2 \times \mathbb{Z}_6$ (choose any generator/s)

2. Fundamental Theorem of Finite Abelian Groups (reading)

Read the first $2\frac{1}{2}$ pages of Chapter 11 (pg 212, 213, and the top half of pg 214).

(1) Write a brief paragraph summarizing a few key ideas for this reading.
(2) Rewrite Theorem 11.1 using more math symbols (currently the theorem uses only words).

3. Fundamental Theorem of Finite Abelian Groups (Computation)

(1) True or false? The group $\mathbb{Z}_{14}$ is isomorphic to the group $\mathbb{Z}_2 \times \mathbb{Z}_7$
(2) True or false? The group $\mathbb{Z}_{16}$ is isomorphic to the group $\mathbb{Z}_4 \times \mathbb{Z}_4$
(3) Which nontrivial direct product is $\mathbb{Z}_{100}$ isomorphic to?
(4) Which nontrivial direct product is $\mathbb{Z}_{12}$ isomorphic to?
(5) How many abelian groups of order $1176 = 2^3 \cdot 3 \cdot 7^2$ are there, up to isomorphism? List all of them.
(6) How many abelian groups of order $540 = 2^2 \cdot 3^3 \cdot 5$ are there, up to isomorphism? List all.

4. Direct products of subgroups of S_7

(1) Let $H = \langle (12), (345) \rangle$ denote the subgroup of S_7 generated by (12) and (345).

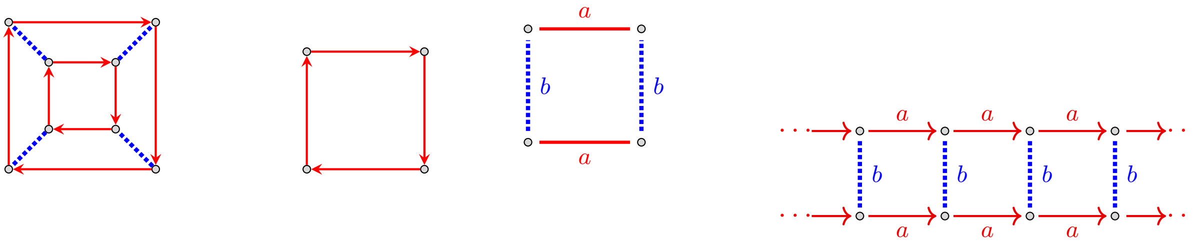
Prove or disprove: There is an isomorphism from $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ onto H.

(2) Let $J = \langle (12345), (67) \rangle$ denote the subgroup of S_7 generated by (67) and (12345).

Prove or disprove: There is an isomorphism from $\mathbb{Z}_{10}$ onto J.

Partial solutions to Question 1

The abstract Cayley diagrams for the first four (G, S) are drawn from left to right below. You should label the vertices with elements of G.

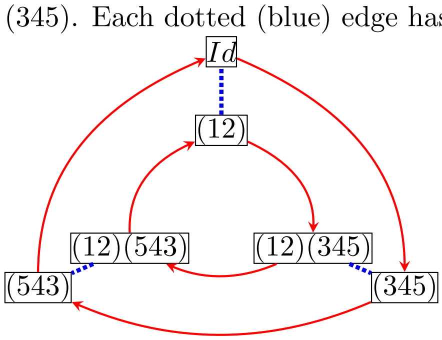


Partial solutions to Question 3:

(1) True because the $gcd(2, 7) = 1$. Alternatively, you can use $(1, 1) \in \mathbb{Z}_2 \times \mathbb{Z}_7$ to generate the entire group and thus showing that it is a cyclic group of order 14.
(2) False because the $gcd(4, 4) = 4$.
Alternatively, you can check that every element in the group $\mathbb{Z}_4$ has order 1, 2, or 4, so every element in $\mathbb{Z}_4 \times \mathbb{Z}_4$ also has order 1, 2, or 4, and thus no element can generate the entire group. Therefore $\mathbb{Z}_4 \times \mathbb{Z}_4$ is not cyclic.
(3) $\mathbb{Z}_{25} \times \mathbb{Z}_4$ (see Theorem 8.2 and its Corollary 2 in Gallian textbook pg 159-160)
(4) N/A
(5) Six. The list of distinct isomorphism classes of abelian groups of order 1176 is given in Gallian Chapter 11 on pg 214.

Solutions to Question 4 part (1)

True.
First, let's find all elements of $H = \langle (12), (345) \rangle$ by drawing its Cayley diagram using the generating set $\{(12), (345)\}$. By definition, H is the set of all products of $(12), (345)$, and their inverses.
Below is the Cayley graph for H with $S = \{(12), (345)\}$ as the generating set. Each solid arrow has label (345). Each dotted (blue) edge has label (12).



We found that $H = \{Id, (12)(345), (543), (12), (345), (12)(543)\}$, which is equal to

$$\{Id, c, c^2, c^3, c^4, c^5\}, \text{ where } c = (12)(345).$$

So H is a cyclic group of order 6. Every cyclic group of order n is isomorphic to $\mathbb{Z}_n$, so H is isomorphic to $\mathbb{Z}_6$ (meaning there exists an isomorphism between H and $\mathbb{Z}_6$).
We will now explicitly define an isomorphism from $\mathbb{Z}_6$ to H. Let $f : \mathbb{Z}_6 \to H$ be defined by

$$f(x) = c^x$$

Below is the Cayley graph for H with $c = (12)(345)$ as the generator (so here the generating set S is the singleton set $\{c\}$). Each solid (red) arrow is labeled by $c = (12)(345)$.