Math 4210/5210 Algebra HW 05. Submit hand-written solutions at the beginning of class.

(Partial solutions to Questions 1, 2, and 4 are given on the next page.)

1. Conjugates

(a) Let $\tau = (123\ldots k)$. Prove that if σ is any permutation, then $\sigma\tau\sigma^{-1} = (\sigma(1)\ \sigma(2)\ \sigma(3)\ \ldots\ \sigma(k))$.

Hint: Note that $\sigma^{-1}(\sigma(i)) = i$. Now compute $\sigma\tau\sigma^{-1}(\sigma(i))$.

(b) Let $\mu = (b_1\, b_2\, \ldots\, b_k)$ be a cycle of length k. Prove that there is a permutation σ such that $\sigma\tau\sigma^{-1} = \mu$.

2. Conjugation computation

(a) Let $\tau = (1234)$ and $\sigma = (135)(724)(89)$. Use Question 1 (and also direct computation) to find $\sigma\tau\sigma^{-1}$.
(b) Let $\sigma = (2587)(93)(46)$. Use Question 1 (and also direct computation) to find $\sigma(123456)\sigma^{-1}$.
(c) Let $\mu = (8275)$. Find a permutation σ such that $\sigma\tau\sigma^{-1} = \mu$.
(d) Find a permutation σ such that $\sigma(123456)\sigma^{-1} = (921754)$.

3. Conjugate subgroups

Fact: Every group of order 4 is either isomorphic to $\mathbb{Z}_4$ or to V_4. Every group of order 3 is isomorphic to $\mathbb{Z}_3$. Below, when asked which group is it isomorphic to, the choices are these three groups.

(a) Let $H = \{Id,\ (1234),\ (13)(24),\ (1432)\}$.
  (i) Which group is H isomorphic to? (Hint: is there an element of order 4?)
  (ii) Draw a Cayley graph for H.
  (iii) Let $a = (387)$. Compute the subgroup aHa^{-1} of S_8, and draw the Cayley graph for aHa^{-1} next to the Cayley graph for H, as shown at the end of class notes (Day 5).
  (iv) Which group is aHa^{-1} isomorphic to?

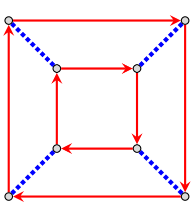
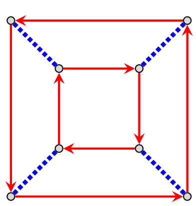
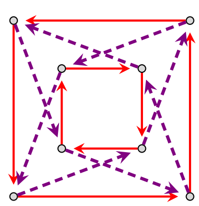
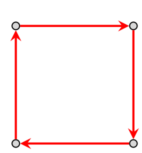
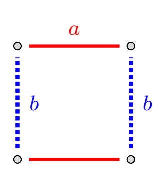
(b) Let $J = \{Id,\ (12)(35),\ (13)(25),\ (15)(23)\}$.
  (i) Which group is J isomorphic to? (Hint: is there an element of order 4?)
  (ii) Draw a Cayley graph for J.
  (iii) Let $a = (387)$. Compute the subgroup aJa^{-1} of S_8, and draw the Cayley graph for aJa^{-1} next to the Cayley graph for J, as shown at the end of class notes (Day 5).
  (iv) Which group is aJa^{-1} isomorphic to?

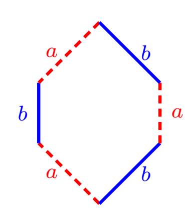
(c) Consider the alternating group A_3
  (i) Which group is A_3 isomorphic to?
  (ii) Draw a Cayley graph for A_3 (as you did in HW04).
  (iii) Let $a = (12)$. Compute $a\,A_3\,a^{-1}$ and draw the Cayley graph for it.
  (iv) What do you notice?

(d) Let R be the counterclockwise rotation by 90^o and let f be your favorite flip in D_4. Let $H = \langle R\rangle = \{R, R^2, R^3, e\}$.
  (i) Draw the Cayley graph for H.
  (ii) What group is H isomorphic to?
  (iii) Compute the group fHf^{-1} and draw the Cayley graph for it.
  (iv) What do you notice?

4. Abelian vs non-abelian Cayley diagrams

(a) What pattern must appear in a Cayley diagram of a group G if and only if G is non-abelian?
(b) Below are Cayley diagrams of six different (pairwise-nonisomorphic) groups. For each Cayley diagram, explain in one-sentence why the corresponding group is abelian or not abelian.



Partial proof for Question 1

(a) Suppose σ is a permutation in S_n. (We need to show that, for all $i = 1, 2, \ldots, k-1$, the permutation $\sigma\tau\sigma^{-1}$ sends $\sigma(i)$ to $\sigma(i+1)$; we also need to show that $\sigma\tau\sigma^{-1}$ sends $\sigma(k)$ to $\sigma(1)$.)

If $i = 1, 2, \ldots, k-1$, then we have

$$\begin{aligned}\sigma\tau\sigma^{-1}(\sigma(i)) &= \sigma\tau(i) \text{ since } \sigma^{-1}\sigma \text{ is the identity map}\\ &= \sigma(i+1) \text{ since } \tau \text{ sends } i \text{ to } i+1\end{aligned}$$

((Now you need to show that $\sigma\tau\sigma^{-1}$ sends $\sigma(k)$ to $\sigma(1)$))

This concludes the proof that $\sigma\tau\sigma^{-1} = (\sigma(1)\ \sigma(2)\ \sigma(3)\ \ldots\ \sigma(k))$.

(b) Since the previous part tells us that $\sigma\tau\sigma^{-1} = (\sigma(1)\ \sigma(2)\ \sigma(3)\ \ldots\ \sigma(k))$, we can choose a permutation σ such that $\sigma(1) = b_1, \sigma(2) = b_2, \ldots, \sigma(k) = b_k$.

Partial answers for Question 2

(a) Let $\tau = (1234)$ and let $\sigma = (135)(724)(89)$. Answer: $\sigma\tau\sigma^{-1} = (135)(724)(89)(1234)(531)(427)(89) = (\sigma(1)\ \sigma(2)\ \sigma(3)\ \sigma(4)) = \boxed{(3457)}$

(b) n/a

(c) Let $\mu = (8275)$. Answer: We need to find a permutation such that

$$\sigma\tau\sigma^{-1} = \mu :$$

$$\sigma(1234)\sigma^{-1} = (8275)$$

We can use the result $\sigma\tau\sigma^{-1} = (\sigma(1)\ \sigma(2)\ \sigma(3)\ \ldots\ \sigma(k))$ from Question 1. Let σ be a permutation which sends 1 to 8, 2 to 2, 3 to 7, 4 to 5. For example, we can choose $\boxed{\sigma = (18)(37)(45)}$ or $\boxed{\sigma = (186)(4537)}$.

Then (if you choose the first boxed option), we have $\sigma\tau\sigma^{-1} = (18)(37)(45)(1234)(18)(37)(45) = (8275) = \mu$

(d) n/a

Partial answers for Question 4:

The pattern on the left *never* appears in the Cayley graph for an abelian group, whereas the pattern on the right illustrates the relation $ab = ba$: