Math 4210/5210 Algebra HW 04
Submit hand-written (or typed) solutions at the beginning of class.

(Partial solutions to Questions 2, 3, and 5 are given on the next page.)

1. Question

Let $n > 1$. Let A_n and B_n denote the set of even permutations and the set of odd permutations in the symmetric group S_n, respectively. Define a map

$$f : A_n \to B_n$$

by

$$f(\pi) = (1\,2)\pi \text{ for all } \pi \in A_n.$$

Prove that this map is injective and surjective. (Thus, the two sets have the same number of elements.)

2. Question

(a) Draw a Cayley diagram for S_3 using generating set $\{(1\ 3),\ (3\ 2\ 1)\}$.
(b) Draw a Cayley diagram for S_3 using generating set $\{(1\ 3),\ (2\ 3)\}$.
(c) What is the alternating group A_3 (of even permutations) in S_3? Draw a Cayley diagram for A_3.

3. Question

Consider the map $\psi : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ defined by

$$\psi(m) = 3m$$

(a) Prove that ψ is a homomorphism.
(b) List the elements in the kernel of ψ.
(c) Is ψ an isomorphism?

4. Question

Let $\mathbb{C}^*$ denote the multiplicative group of nonzero complex numbers.

Consider the map $\phi : \mathbb{C}^* \to \mathbb{C}^*$ defined by

$$\phi(z) = z^4$$

(a) Prove that ϕ is a homomorphism.

(b) List the elements in the kernel of ϕ.
Note: Since 1 is the identity element in $\mathbb{C}^*$, the kernel of ϕ is $\ker \phi = \{z \in \mathbb{C}^* : \phi(z) = 1\}$.

(c) Is ϕ an isomorphism? (If yes, prove that it is both surjective *and* injective; if no, prove that it's not injective *or* not surjective.)

(d) Find an element of order 2 in $\mathbb{C}^*$. Find two elements of order 4 in $\mathbb{C}^*$.

5. Inner automorphism

Warm-up: First, study the 4-step proof (that the conjugation map is an isomorphism) in Example 6 of Chapter 6 (pg 123).

(a) Warm-up: Rewrite this same proof in your own words (or exactly as written in the book).
(b) Prove that the function ϕ_a defined on pg 128 is indeed an automorphism. Mimic the four steps given in Example 6.
(c) What is the kernel of ϕ_a? Why?
(d) Consider the cyclic subgroup $H = \langle(1234)\rangle$ of S_4. List all elements in H, and list all elements of the subgroup $(23)H(23)^{-1}$. Draw the Cayley diagrams for H and $(23)H(23)^{-1}$ in such a way that we can see that the two groups are isomorphic.

Answers for Question 2:

(a) Answer: Same graph as given in class notes for $S = \{(12), (123)\}$ but with vertices arranged differently.
(b) Answer: Same graph as given in class notes for $S = \{(12), (23)\}$ but with vertices arranged differently.
(c) Answer: $A_3 = \{(1), (123), (321)\} = \langle(123)\rangle = \langle(321)\rangle$ is a cyclic group of order 3, so your Cayley diagram should look like a Cayley diagram for $\mathbb{Z}_3$ with generator 1.

Answers for Question 3:

(a) *Proof.* Suppose $a, b \in \mathbb{Z}_{12}$. Then $\psi(a + b) = 3(a + b) = 3a + 3b = \psi(a) + \psi(b)$. □
(b) *Answer:* The elements in $\ker\psi$ are the elements $a \in \mathbb{Z}_{12}$ such that $3a$ is congruent to 0 modulo 12, that is, $3a - 0$ is divisible by 12. So

$$\ker\psi = \{0, 4, 8\}$$

□
(c) *Answer:* No, the function ψ is not an isomorphism. For example, we know ψ is not injective since $\psi(0) = 0 = \psi(4)$ although $0 \neq 4$ in $\mathbb{Z}_{12}$. □

Hint for Question 5:

For part (d), the Cayley diagrams for both subgroups should be a cycle of length 4.