MATH 4210/5210 ALGEBRA HW01

SUBMIT HAND-WRITTEN (OR TYPED) SOLUTIONS AT THE BEGINNING OF CLASS

Note: Be prepared to present your proofs to other students and the instructor during class; be prepared to listen to others' explanation. The presenters can use notes (and the listeners can take notes). It's okay if your proof is imperfect or incorrect. The presenter will be graded by preparation and effort; the audience member will be graded by engagement.

0. Cut-out polygons

Cut out an equilateral triangle, a square, and a rectangle. (You can use these cut-outs during the quiz!)

1. Definitions

From Chapter 2, pg 42-43:

- Write (and memorize) the definition of a binary operation on a set A
- Write (and memorize) the definition of an associative binary operation
- Write (and memorize) the definition of an identity element for a binary operation
- Write (and memorize) the definition of a group
- Write (and memorize) the definition of an inverse of an element x in a group G

2. Odd integers

Consider the set of odd integers, under addition.

- Is is a group? (Prove or disprove)
- Is addition a binary operation on the set of odd integers? (Prove or disprove)

3. Right and left cancellation

Let G be a group, and let $a, b, c \in G$.

(i) (Right cancellation) Prove that $ba = ca$ implies $b = c$.
(ii) (Left cancellation) Prove that $ab = ac$ implies $b = c$.

Proof. (i) (Instructor's proof)
Suppose $ba = ca$. Multiply on the right by a^{-1}:

$$(ba)a^{-1} = (ca)a^{-1}$$

Associativity give us

$$b(aa^{-1}) = a(a^{-1})$$
$$be = ce$$
$$b = c$$

(ii) (Insert your proof)

□

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4. Uniqueness of inverses

Prove that each element of a group G has a unique inverse.

(Hint: You can assume Question 3 or Theorem 2.2 in Chapter 2. Follow the proof in the book for Theorem 2.3 or write your own.)

5. Square mattress group D_4

Consider the "Mattress Group" (see Day 1 class notes). Since the mattress is in a non-square rectangle bed frame, there are exactly four transformations that can be done to the mattress (the identity, the 180^o rotation, the horizontal flip, and the vertical flip). Observe that $t_1 t_2 = t_2 t_1$ for *all* transformations t_1, t_2.

Now, suppose your mattress is a square mattress in a square bed frame.

(i) Describe all transformations that can be done to this square mattress. (Hint: This mattress group is "the same" as D_4, the *symmetry group of a regular n-gon*. See description of D_4 given in Chapter 1, pg 31-33)

(ii) Give an example of two transformations t_1, t_2 such that $t_1 t_2 \neq t_2 t_1$. (This means the "Square mattress" group is *not* abelian).

(iii) Give an example of two transformations t_1, t_2 such that $t_1 t_2 = t_2 t_1$. (Although you can find two elements that do commute, this group is still *not* abelian because of part (ii).)

6. An abelian group

Let $S = \mathbb{R}\backslash\{-1\}$. Define a binary operation on S by $a \star b = a + b + ab$. Show that $(S, \star)$ is an abelian group.

(i) Prove that this is indeed a binary operation on S, that is, show that if $a, b \in S$ then $a \star b \in S$

(ii) Prove that $\star$ is associative

(iii) Prove that S contains the identity of $\star$

(iv) Prove that every element in S has an inverse under $\star$

(v) Prove that $\star$ is commutative

Proof. (i) (Instructor's proof)

For the sake of contradiction, suppose that S is not closed under $\star$. So there exist $a, b \in S$ such that $a \star b \notin S$, that is, $a \star b = -1$. Then

$$-1 = a + b + ab$$

So we have

$$\begin{aligned} 0 &= 1 + a + b + ab \\ &= (1 + a) + b(1 + a) \\ &= (1 + b)(1 + a) \end{aligned}$$

Therefore, either $1 + b = 0$ or $1 + a = 0$. This implies $b = -1$ or $a = -1$, which contradicts the fact that $a, b \in S$. Hence S is closed under $\star$.

(ii) (Hint: Prove that $(a \star b) \star c = a \star (b \star c)$ for all $a, b, c \in S$)

(iii) (Hint: What is the identity e for $\star$? Prove that $e \star a = a = a \star e$ for all $a \in S$.)

(iv) (Hint: Give a formula for the inverse of each $a \in S$ under $\star$, and prove that this element is also in S.)

(v) (Why is $(S, \star)$ abelian?)

$\square$