1. For each statement below, determine if it is true or false. Prove your answer.

(a) If the order of a group $G$ is infinite (that is, if there are infinitely many elements in $G$), then the order of every non-identity $x \in G$ is also infinite.

> **Solution:** False. The multiplicative group $\mathbb{R}^*$ is infinite, but the element $-1$ has order 2.

(b) Every cyclic group is abelian.

> **Solution:** True.
> Possible explanation 1: Every cyclic group is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}_n$.
> Possible explanation 2: A cyclic group $G$ is a group which can be generated by only one element, so $G = \langle r \rangle$ for some $r \in G$. If $x, y \in G$, then $x = r^k$ and $y = r^\ell$ for some $k, \ell \in \mathbb{Z}$. So $xy = r^k r^\ell = r^{k+\ell} = r^\ell r^k = yx$.

(c) Every abelian group is cyclic.

> **Solution:** False. Proof: A possible counterexample is $V_4$ (i.e. the rectangle mattress group) which is not cyclic (since it's a group of order 4 which is not isomorphic to $\mathbb{Z}_4$). It requires at least two generators.

(d) Every dihedral group is abelian.

> **Solution:** False. Proof: The dihedral group $D_3$ of order 6 is not abelian, for example, rotation by $120^o$ followed by a flip is not the same as the same flip followed by a rotation by $120^o$.

(e) Every symmetric group is not abelian.

> **Solution:** False. The symmetric group $S_2 = \{Id,\ (12)\}$ on two objects is cyclic and therefore abelian.
> However, the statement "Every symmetric group $S_n$ is not abelian for $n \geq 3$" is a true statement.

(f) There is a cyclic group of order 100.

> **Solution:** True. Proof: Take the additive group $\mathbb{Z}_{100}$, or the multiplicative subgroup of the 100-th roots of unity in $\mathbb{C}^*$.

(g) There is a symmetric group of order 100

> **Solution:** False. The number 100 is not equal to any factorial. Check that $4! = 24 < 100 < 5! = 120$.

(h) If some pair of distinct, non-identity elements in a group commute, then the group is abelian.

> **Solution:** False. In $D_3$, the elements $R$ and $R^2$ commute, but $D_3$ is not abelian.

(i) If every pair of elements in a group commute, the group is cyclic.

> **Solution:** False. The group $V_4$ is not cyclic, but every pair of elements commutes.

(j) If every pair of elements in a group commute, the group is abelian.

**Solution:** True, by definition.

2. (a) Is there a dihedral group of order 27?

**Solution:** No. The dihedral group $D_n$ has $n$ reflections and $n$ rotations (for some positive integer $n$), so the order of a dihedral group is even.

(b) If an alternating group $A_n$ has order $M$, what order does the symmetric group $S_n$ have?

**Solution:** The order of $S_n$ is $2M$, since we've seen that there is a bijection between the set of even permutations and the set of odd permutations and even permutations of $S_n$.

3. For each part below, compute the orbit of the element in the group.

(a) The element $R^2$ in the group $D_{10}$

**Solution:** $\langle R^2 \rangle = \{R^2, R^4, R^6, R^8, e\}$

(b) The element 10 in $\mathbb{Z}_{16}$

**Solution:** $\langle 10 \rangle = \{10, 4, 14, 8, 2, 12, 6, 0\}$

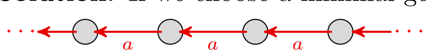
(c) The element 25 in the group $\mathbb{Z}_{30}$

**Solution:** $\langle 25 \rangle = \{25, 20, 15, 10, 5, 0\}$

4. Recall that $\mathbb{Z}$ is a group under the operation of ordinary addition.

(a) Create a Cayley diagram for it.

**Solution:** If we choose a minimal generating set $\{1\}$, we have the following (where $a = +1$):




(b) Is it abelian?

**Solution:** Yes, it is a cyclic group, since it can be generated by the element 1 or $-1$.

(c) Give a minimal generating set consisting of more than one element.

**Solution:** For example, $\{2, 3\}$ or $\{7, 12\}$ would work.

5. (a) Is there a group (of order larger than 1) in which no element (other than the identity) is its own inverse?

**Solution:** Yes. For example, the cyclic group of order 3. You can observe this from the multiplication table.

(b) Is there a group (of order larger than 3) in which no element (other than the identity) is its own inverse?

**Solution:** Yes. For example, the cyclic group $\mathbb{Z}_5$. Every non-identity element has order 5.

(c) Find a group (of order larger than 1) such that there is only one solution to the equation $x^2 = e$, that is, the solution $x = e$, or explain why no such group exists.

**Solution:** The groups in the solutions to parts (a), (b) would work.

(d) Find a group that has exactly two solutions to the equation $x^2 = e$, or explain why no such group exists.

**Solution:** The cyclic group of order 4 generated by $r$. The two solutions are $x = e$ and $x = r^2$.

(e) Find a group with more than 2 solutions to the equation $x^2 = e$, or explain why no such group exists.

**Solution:** The Klein-4 group $\langle a, b\rangle$ with minimal generating set $\{a, b\}$. There are four solutions, $x = e, x = a, x = b$, and $x = ab$. You can observe this from the multiplication table, or consider the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ and check that all four elements satisfy the equation $x^2 = e$.

(f) Find a group with at least two elements in it, and only one solution to the equation $x^3 = e$ (that is, the solution $x = e$) or explain why no such group exists.

**Solution:** The groups $\mathbb{Z}_2, \mathbb{Z}_4$, and $V_4$ would work.

(g) Find a group that has more than two solutions to the equation $x^3 = e$, or explain why no such group exists.

**Solution:** In the cyclic group $\mathbb{Z}_3$, every element satisfies the equation $x^3 = e$.

(h) You have seen two non-isomorphic groups of order 6. What are their names? Specify which, if any, are abelian.

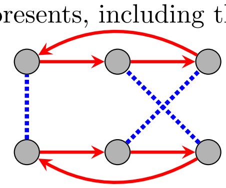
**Solution:** One is non-abelian, the Dihedral group $D_3$, which is isomorphic to the symmetric group $S_3$. The other is the cyclic group $\mathbb{Z}_6$, which is abelian.

(i) Suppose $m$ is a positive integer. If there exists only one group of order $m$, to what family must this group belong? Why?

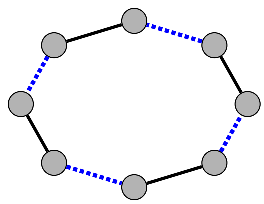
**Solution:** For each positive integer $k$, we have the cyclic group $\mathbb{Z}_k$ is a group. Since there exists only one group of order $m$, this group must belong to the family of cyclic groups.

6. Determine whether each of the following diagrams are Cayley diagrams. If the answer is "yes," say what familiar group it represents, including the generating set. If the answer is "no," explain why.

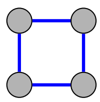
(a)



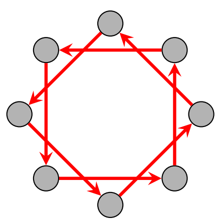
**Solution:** Yes.
This is the Cayley diagram of $D_3$ with generating set $\{R, f\}$, where $R$ is a rotation by $2\pi/3$ and $f$ is any flip.
It could also be the Cayley diagram of $S_3$ with generating set $\{(123), (12)\}$.
Compare this Cayley diagram with the Cayley diagram in class notes (where the dashed edges don't cross). It's the same diagram, only with the vertices rearranged.

(b) 

**Solution:** Yes. This is the Cayley diagram of the Dihedral group $D_4$ with minimal generating set $f, g$, where $f$ and $g$ are reflections with respect to mirrors $45^o$ apart.

(c) 

**Solution:** No. There is only one type of arrow, which means that there is only one generator. This arrow is double-sided, which means that this generator is of order 2. If this is the Cayley diagram of a group, the group should have order 2, not 4.

(d) 

**Solution:** No. There is only one type of arrow, which means that there is only one generator. This arrow has order 4 because we see that four arrows form a 4-cycle. If this is the Cayley diagram of a group, the group should have order 4, not 8.

7. Answer the following questions about permutations and the symmetric group.

   (a) Write as a product of disjoint cycles (read from right to left as usual):
   $(1\ 5\ 2)\,(1\ 2\ 3\ 4)\,(1\ 3\ 5) =$
   $(1\ 3\ 5)\,(1\ 2\ 3\ 4)\,(1\ 5\ 2) =$

   **Solution:** $(1\ 5\ 2)\,(1\ 2\ 3\ 4)\,(1\ 3\ 5) = (1\ 4\ 5)(2\ 3)$
   $(1\ 3\ 5)\,(1\ 2\ 3\ 4)\,(1\ 5\ 2) = (1)(2)(3\ 4)(5) = (3\ 4)$

   (b) Write $(1\ 2\ 3\ 4)$ as a product of *transpositions* (i.e., 2-cycles). Read from right to left as usual.

   **Solution:** (14)(13)(12) or (12)(24)(23) or (23)(31)(34) or (34)(24)(14) (or other options)

   (c) Gallian textbook Chapter 5 Exercise 8(a)–(e) on page 113. Determine whether given permutations are even or odd.

**Solution:**

(a) even

(b) odd

(c) even

(d) odd (=odd+even+even)

(e) even (= odd + odd)

(d) What is the *inverse* of the element (1 3 2 6) (4 5) in $S_6$?

**Solution:** (45)(1623)

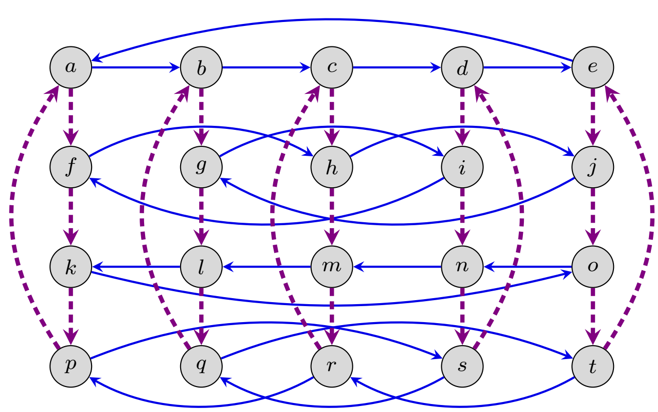
(e) The *order* of an element $g \in G$ is equal to the order (number of elements) of $\langle g \rangle$, the group generated by $g$. When the order is finite, it is also the minimum positive integer $k$ such that $g^k = e$. What is the order of the element (1 2 3 6) (4 5 7) in $S_7$?

**Solution:** The order is 12 since $[(1\ 2\ 3\ 6)\,(4\ 5\ 7)]^i \neq id$ for $i = 1, 2, ..., 11$ and $[(1\ 2\ 3\ 6)\,(4\ 5\ 7)]^{12} = id$. Alternatively, use Theorem 5.3 (pg 100) which says that the order of a permutation $\pi$ is the lcm of the lengths of the cycles of $\pi$ (when $\pi$ is written in cycle notation).

(f) Find an element of order 20 in $S_9$.

**Solution:** (1 2 3 4 5) (6 7 8 9)

8. Let $G$ be the group whose Cayley diagram is shown below, and suppose $e$ is the identity element. Consider the subgroups $A = \langle a \rangle = \{a, b, c, d, e\}$ and $J = \langle j \rangle = \{e, j, o, t\}$.




Use this Cayley diagram as a "group calculator". Start at the identity element, then chase the sequence through the Cayley graph,

What is the letter that represents the group element $j^3a$?

**Solution:** Answer: $r$

What is the letter that represents the group element $a^2j^2aj$?

**Solution:** Answer: $p$

What is the letter that represents the group element $jaj^{-1}$?

**Solution:** Answer: $b$

What is the letter that represents the group element $a^{-2}ja$?

**Solution:** Answer: $j$

9. The *center* of a group $G$ is the set

$$Z(G) = \{z \in G \mid gz = zg, \text{ for all } g \in G\} = \{z \in G \mid gzg^{-1} = z, \text{ for all } g \in G\}.$$

It is a subgroup of $G$.

a. Compute the center of $\mathbb{Z}_6$.

**Solution:** $\mathbb{Z}_6$ is abelian, so the entire group is the center.

b. Compute the center of $D_4$.

**Solution:** The center of $D_4$ is $\langle R^2 \rangle$. Reason: the half circle rotation commutes with every reflection (and every rotation). A different rotation does not commute with a reflection (for example, f). None of the reflections commutes with $R$.

c. Compute the center of $D_5$.

**Solution:** The center of $D_5$ is the trivial group. Reason: None of the rotations commutes with $f$. None of the reflections commutes with $R$.

d. Consider the group $A_3$ of even permutations. Compute the center of $A_3$.

**Solution:** $A_3$ is abelian, and therefore the center of $A_3$ is the entire group $A_3$.
To see why $A_3$ is abelian, notice that $A_3$ is a cyclic group of order 3, since it can be generated by the 3-cycle (123).
Another way to see that $A_3$ is abelian, is to compute its order which is $3!/2 = 3$. We've seen that every group of order 3 (or any prime number) is cyclic.

e. Consider the group $A_n$ of even permutations, where $n \geq 4$. Prove that (1 2 3) is not in the center of $A_n$ by producing another even permutation which does not commute with (1 2 3).

**Solution:** The element (2 3 4) works. (2 3 4)(1 2 3) = (12)(34)
(1 2 3)(2 3 4) = (13)(24)

f. Let $n \geq 4$. Prove that $(1\ 2)(3\ 4)$ is not in the center of $A_n$.

> **Solution:** For example, you can show that the element $(1\ 2\ 3)$ does not commute with $(1\ 2)(3\ 4)$.

g. Compute the center of $A_4$
Hint: A non-identity permutation in $S_4$ is an even permutation if and only of its cycle notation is of the form $(ab)(cd)$ or $(abc)$. (Make sure you can prove this!)
Do $(ab)(cd)$ and $(abc)$ commute?

> **Solution:** Answer: The answer is the trivial group.
> Reason: The permutations $(abc)$ and $(ab)(cd)$ do not commute.
> $(abc)(ab)(cd) = (a)(bdc)$ and $(ab)(cd)(abc) = (acd)(b)$.

h. Compute the center of $S_4$.
Hint: Every non-identity permutation in $S_4$ can be written in the form $(ab)$, $(abc)$, $(abcd)$, and $(ab)(cd)$. Can you find a permutation that does not commute with $(ab)$? With $(abcd)$?

i. Compute the center of $S_2$.

> **Solution:** This group is abelian, so the center is the entire group.

10. Notation/Definition: Let $G$ be a group and $x \in G$.

- The *conjugacy* class of $x$ is the set $\mathrm{cl}_G(x) := \{gxg^{-1} \mid g \in G\}$.
- Let $Z(G)$ be the set $\{z \in G \mid gz = zg \text{ for all } g \in G\}$.

Prove that $\mathrm{cl}_G(x) = \{x\}$ if and only if $x \in Z(G)$.

11. You can use the following fact.

**Proposition 1.** For any $\sigma \in S_n$, we have $\sigma\ (a_1\ a_2\ \dots\ a_k)\ \sigma^{-1} = (\,\sigma(a_1)\ \sigma(a_2)\ \dots\ \sigma(a_k)\,)$.

(a) Let $x$ be a $k$-cycle. Prove that $y \in S_n$ is conjugate to $x$ iff $y$ is a $k$-cycle.

> **Solution:** By Proposition 1, every pair of $k$-cycles are conjugate.

(b) Prove that the permutations $(12)$ and $(14)$ in $S_6$ are conjugate by finding a permutation $p \in S_6$ such that $p^{-1}(12)p = (14)$.

(c) List all permutations in $S_4$ which are conjugate to $(1234)$. Use the fact from part (a).

> **Solution:** The answer is $(1234), (1432), (1243), (1342), (1324), (1423)$. Explanation: The permutations which are conjugate to $(1234)$ in $S_4$ are all the 4-cycles.

12. Let $\phi : (\mathbb{Z}, +) \to (\mathbb{Z}, +)$ be the map given by $\phi(n) = 7n$ for $n \in \mathbb{Z}$. Find the kernel and the image of $\phi$.

> **Solution:** The kernel of $\phi$ is the trivial subgroup $\{0\}$. The image of $\phi$ is $7\mathbb{Z}$, the subgroup of all integer multiples of 7.

13. Consider the group homomorphism $f : (\mathbb{R}, +) \to (\mathbb{C}^*, \times)$ defined by

$$f(\theta) = \cos\theta + i\sin\theta.$$

(a) Find the kernel of $f$ and the image of $f$.

**Solution:** The kernel is the subgroup $\{2\pi k : k \in \mathbb{Z}\}$ of $(\mathbb{R}, +)$. The image is the circle subgroup $\{x \in \mathbb{C}^* : |x| = 1\} = \{a + ib \in \mathbb{C}^* : \sqrt{a^2 + b^2} = 1\}$ of $\mathbb{C}^*$.

(b) Give an isomorphism (bijective group homomorphism) from the kernel of $f$ to $(\mathbb{Z}, +)$.

**Solution:** Let $f$ send each $2\pi k \in \ker f$ to $k \in \mathbb{Z}$.

14. Let $G$ be a group and let $g$ be some element in $G$. Consider the group homomorphism $f : \mathbb{Z} \to G$ given by

$$f(n) = g^n.$$

(a) If the order of $g$ is infinite, what is the kernel of $f$? Justify.

**Solution:** The kernel is the trivial subgroup $\{0\}$ of $\mathbb{Z}$.

(b) If the order of $g$ is finite, say $m$, what is the kernel of $f$? Justify.

**Solution:** The kernel is the subgroup $m\mathbb{Z} = \{mk : k \in \mathbb{Z}\}$ of $\mathbb{Z}$.