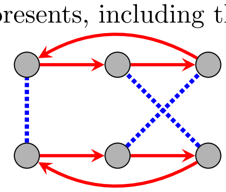
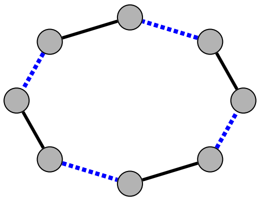
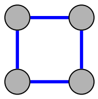
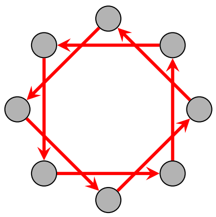
1. For each statement below, determine if it is true or false. Prove your answer.
   (a) If the order of a group G is infinite (that is, if there are infinitely many elements in G), then the order of every non-identity $x \in G$ is also infinite.
   (b) Every cyclic group is abelian.
   (c) Every abelian group is cyclic.
   (d) Every dihedral group is abelian.
   (e) Every symmetric group is not abelian.
   (f) There is a cyclic group of order 100.
   (g) There is a symmetric group of order 100
   (h) If some pair of distinct, non-identity elements in a group commute, then the group is abelian.
   (i) If every pair of elements in a group commute, the group is cyclic.
   (j) If every pair of elements in a group commute, the group is abelian.
2. (a) Is there a dihedral group of order 27?
   (b) If an alternating group A_n has order M, what order does the symmetric group S_n have?
3. For each part below, compute the orbit of the element in the group.
   (a) The element R^2 in the group D_{10}
   (b) The element 10 in $\mathbb{Z}_{16}$
   (c) The element 25 in the group $\mathbb{Z}_{30}$
4. Recall that $\mathbb{Z}$ is a group under the operation of ordinary addition.
   (a) Create a Cayley diagram for it.
   (b) Is it abelian?
   (c) Give a minimal generating set consisting of more than one element.
5. (a) Is there a group (of order larger than 1) in which no element (other than the identity) is its own inverse?
   (b) Is there a group (of order larger than 3) in which no element (other than the identity) is its own inverse?
   (c) Find a group (of order larger than 1) such that there is only one solution to the equation $x^2 = e$, that is, the solution $x = e$, or explain why no such group exists.
   (d) Find a group that has exactly two solutions to the equation $x^2 = e$, or explain why no such group exists.
   (e) Find a group with more than 2 solutions to the equation $x^2 = e$, or explain why no such group exists.
   (f) Find a group with at least two elements in it, and only one solution to the equation $x^3 = e$ (that is, the solution $x = e$) or explain why no such group exists.
   (g) Find a group that has more than two solutions to the equation $x^3 = e$, or explain why no such group exists.
   (h) You have seen two non-isomorphic groups of order 6. What are their names? Specify which, if any, are abelian.
   (i) Suppose m is a positive integer. If there exists only one group of order m, to what family must this group belong? Why?
6. Determine whether each of the following diagrams are Cayley diagrams. If the answer is "yes," say what familiar group it represents, including the generating set. If the answer is "no," explain why.
   (a) 

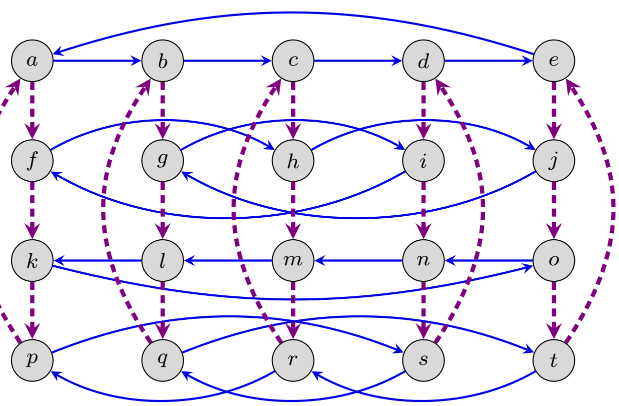
(b) 

(c) 

(d) 

7. Answer the following questions about permutations and the symmetric group.
   (a) Write as a product of disjoint cycles (read from right to left as usual):
   $(1\ 5\ 2)\,(1\ 2\ 3\ 4)\,(1\ 3\ 5) =$
   $(1\ 3\ 5)\,(1\ 2\ 3\ 4)\,(1\ 5\ 2) =$
   (b) Write $(1\ 2\ 3\ 4)$ as a product of *transpositions* (i.e., 2-cycles). Read from right to left as usual.
   (c) Gallian textbook Chapter 5 Exercise 8(a)–(e) on page 113. Determine whether given permutations are even or odd.
   (d) What is the *inverse* of the element $(1\ 3\ 2\ 6)\,(4\ 5)$ in S_6?
   (e) The *order* of an element $g \in G$ is equal to the order (number of elements) of $\langle g \rangle$, the group generated by g. When the order is finite, it is also the minimum positive integer k such that $g^k = e$. What is the order of the element $(1\ 2\ 3\ 6)\,(4\ 5\ 7)$ in S_7?
   (f) Find an element of order 20 in S_9.

8. Let G be the group whose Cayley diagram is shown below, and suppose e is the identity element. Consider the subgroups $A = \langle a \rangle = \{a, b, c, d, e\}$ and $J = \langle j \rangle = \{e, j, o, t\}$.



Use this Cayley diagram as a "group calculator". Start at the identity element, then chase the sequence through the Cayley graph,

What is the letter that represents the group element j^3a?

What is the letter that represents the group element a^2j^2aj?

What is the letter that represents the group element jaj^{-1}?

What is the letter that represents the group element $a^{-2}ja$?

9. The *center* of a group G is the set

$$Z(G) = \{z \in G \mid gz = zg, \text{ for all } g \in G\} = \{z \in G \mid gzg^{-1} = z, \text{ for all } g \in G\}.$$

It is a subgroup of G.

a. Compute the center of $\mathbb{Z}_6$.
b. Compute the center of D_4.
c. Compute the center of D_5.
d. Consider the group A_3 of even permutations. Compute the center of A_3.
e. Consider the group A_n of even permutations, where $n \geq 4$. Prove that $(1\ 2\ 3)$ is not in the center of A_n by producing another even permutation which does not commute with $(1\ 2\ 3)$.
f. Let $n \geq 4$. Prove that $(1\ 2)(3\ 4)$ is not in the center of A_n.
g. Compute the center of A_4
Hint: A non-identity permutation in S_4 is an even permutation if and only of its cycle notation is of the form $(ab)(cd)$ or (abc). (Make sure you can prove this!)
Do $(ab)(cd)$ and (abc) commute?
h. Compute the center of S_4.
Hint: Every non-identity permutation in S_4 can be written in the form (ab), (abc), $(abcd)$, and $(ab)(cd)$. Can you find a permutation that does not commute with (ab)? With $(abcd)$?
i. Compute the center of S_2.

10. Notation/Definition: Let G be a group and $x \in G$.

- The *conjugacy* class of x is the set $\text{cl}_G(x) := \{gxg^{-1} \mid g \in G\}$.
- Let $Z(G)$ be the set $\{z \in G \mid gz = zg \text{ for all } g \in G\}$.

Prove that $\text{cl}_G(x) = \{x\}$ if and only if $x \in Z(G)$.

11. You can use the following fact.

Proposition 1. For any $\sigma \in S_n$, we have $\sigma\ (a_1\ a_2\ \ldots\ a_k)\ \sigma^{-1} = (\,\sigma(a_1)\ \sigma(a_2)\ \ldots\ \sigma(a_k)\,)$.

(a) Let x be a k-cycle. Prove that $y \in S_n$ is conjugate to x iff y is a k-cycle.
(b) Prove that the permutations (12) and (14) in S_6 are conjugate by finding a permutation $p \in S_6$ such that $p^{-1}(12)p = (14)$.
(c) List all permutations in S_4 which are conjugate to (1234). Use the fact from part (a).

12. Let $\phi : (\mathbb{Z}, +) \to (\mathbb{Z}, +)$ be the map given by $\phi(n) = 7n$ for $n \in \mathbb{Z}$. Find the kernel and the image of ϕ.

13. Consider the group homomorphism $f : (\mathbb{R}, +) \to (\mathbb{C}^*, \times)$ defined by

$$f(\theta) = \cos\theta + i\sin\theta.$$

(a) Find the kernel of f and the image of f.
(b) Give an isomorphism (bijective group homomorphism) from the kernel of f to $(\mathbb{Z}, +)$.

14. Let G be a group and let g be some element in G. Consider the group homomorphism $f : \mathbb{Z} \to G$ given by

$$f(n) = g^n.$$

(a) If the order of g is infinite, what is the kernel of f? Justify.
(b) If the order of g is finite, say m, what is the kernel of f? Justify.