1. (a) Let $n > 1$. Let A_n and B_n denote the set of even permutations and the set of odd permutations, respectively. Define a map $f : A_n \to B_n$ by $f(\pi) = (1\,2)\pi$ for all $\pi \in A_n$.
Prove that this map is injective and surjective.

Solution: A similar proof is given in the proof of Theorem 5.7 on pg 105.

(b) Let H be a subgroup of a group G, and let $x \in G$. Define a bijective map f from H to xH.

Solution: Define
$$f\colon H \longrightarrow xH\,, \qquad \text{by } f(h) = xh$$
for all $h \in H$.

(c) Show that this map is surjective.

Solution: Suppose $b \in xH$. Then by definition of left coset, $b = xh$ for some $h \in H$. Let $a := h$. Then $f(a) = xa = xh = b$, as needed.

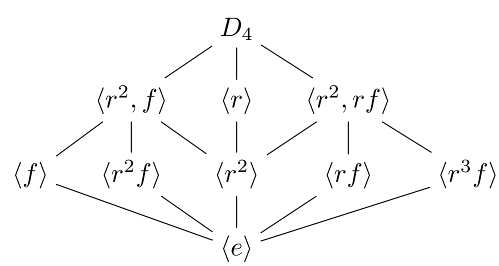
(d) Suppose G is a non-abelian group of order 1000 and H is a subgroup of order 20. Let x be an element of G which is not in H.
(i) How many elements are in the left coset xH?
(ii) How many elements are in the right coset Hx?

Solution: (i-ii)The size of every left coset (and also right coset) is the same as the size of H, so the answer is 20 for both questions.

(iii) How many left cosets of H are there?

Solution: (iii) By Lagrange's Theorem, there are $1000/20 = 50$ left cosets of H.

2. (a) I listed all subgroups of D_4 (in a subgroup lattice) below. Label each edge between $K \leq H$ with the index $[H : K]$.



Solution: The label on each edge is 2. This is because the order is 8 for the subgroup at the top level, 4 for the subgroup at the 2nd highest level, 2 at the third highest level.

(b) Is $f\langle r\rangle = \langle r\rangle f$? What about other left and right cosets of $\langle r\rangle$? Prove your answer.

Solution: Yes, $x\langle r\rangle = \langle r\rangle x$ for all $x \in D_4$. First, we see that the group $\langle r\rangle$ has order 4. We know that the group D_4 has order 8. By Lagrange's theorem, we get that $[D_4 : \langle r\rangle] = 8/4 = 2$. We've seen in class that this implies that the left cosets of $\langle r\rangle$ and the right cosets of $\langle r\rangle$ coincide.

(c) Is the left coset $r^3 f\langle r^2, f\rangle$ equal to the right coset $\langle r^2, f\rangle r^3 f$?

> **Solution:** Yes. Similar explanation as the previous part.

3. (a) If H is a subgroup of G and $a \in G$, then a left coset aH is ... [give the definition]

> **Solution:** the set $\{ah : h \in H\}$

(b) The *index* $[G : H]$ of a subgroup $H \leq G$ is [give a definition, not a theorem!] ...

> **Solution:** ... the number of left cosets of H.

Theorem 1. Let H be a subgroup of G. Then the following are all equivalent.

(i) The subgroup H is called *normal* in G, that is, $gH = Hg$ for all $g \in G$; ("left cosets are right cosets");

(ii) $ghg^{-1} \in H$ for all $h \in H, g \in G$; ("closed under conjugation").

(iii) $gHg^{-1} = H$ for all $g \in G$; ("only one conjugate subgroup")

---

4. (a) Consider the subgroup $H = \{(1), (1, 2)\}$ of S_3. Is H normal?

> **Solution:** No, you can check that $(123)H$ is not equal to $H(123)$.
>
> Another example that would work is $(13)H \neq H(13)$.
> A possibly faster way to determine this is to see that $(13) = (23)(12)(23)^{-1}$ and $(23) = (13)(12)(13)^{-1}$ are conjugate to (12), but they are not in H, hence failing part (ii) of the above theorem for being normal.

(b) Consider the subgroup $J = \{(1), (123), (132)\}$ of S_3. Is J normal?

> **Solution:** Yes, there is only other left coset of J (other than J itself), and there is only other other right coset of J (other than J), so they must be the same.
> This satisfies part (i) of the above theorem, Theorem 1, for being normal.

(c) Consider the subgroup $H = \langle(1234)\rangle$ of S_4. Is H normal?

> **Solution:** No. For example, the 4-cycle (1324) is a conjugate of (1234) but it is not in H.

(d) Let $n > 2$. Is A_n a normal subgroup of S_n?

> **Solution:** Yes. Proof: There are exactly two left cosets of A_n in S_n. So the left coset xA_n which is not equal to A_n must equal the right coset which is not equal to A_n.

(e) Consider a mystery subgroup K of $\mathbb{Z}_5 \times \mathbb{Z}_8$. Is K normal?

> **Solution:** Every subgroup of an abelian group is normal, so K is normal.

5. Let H be a subgroup of G. Given two fixed elements $a, b \in G$, define the sets

$$aHbH := \{ah_1bh_2 : h_1, h_2 \in H\} \qquad \text{and} \qquad abH := \{abh : h \in H\}.$$

(a) Prove that *if* H is normal then $aHbH \subset abH$.

> **Solution:** To show $aHbH \subset abH$, let $h_1, h_2 \in H$. We need to show that ah_1bh_2 can be written as abh for some $h \in H$. Since H is normal in G, the left coset bH is equal to the right coset Hb. Hence we can write h_1b as bh_3 for some $h_3 \in H$, so $ah_1bh_2 = abh_3h_2$, which is in abH since $h_3h_2 \in H$.

(b) Prove that the statement is false if we remove the "normal" assumption. That is, give a specific G and H and $a, b \in G$ such that $aHbH$ is not a subset of abH.

> **Solution:** Possible proof: Let $G = D_3$, let $H = \langle f \rangle$. But $rfre = rfr = f$, which is in $rHrH$ but not in $r^2H = \{r^2, r^2f\}$, so $rHrH \neq r^2H$.
>
> Try to come up with a similar proof but using S_3.
>
> Possible scratch work (thought process):
> Let $G = D_3$ (because every group with order 5 or lower is abelian). To come up with a counterexample, I have to make sure to pick a non-normal subgroup H (since the statement is true if H is a normal subgroup), so I can pick one of the subgroups which is generated by exactly one reflection, $\langle f \rangle$ or $\langle rf \rangle$ or $\langle r^2f \rangle$.
>
> I pick $H := \{e, f\}$. To come up with a counterexample, I have to make sure to pick $a, b \notin H$ (otherwise the statement would be true).
>
> First, I try $a = r$ and $b = r$, and I check whether $aHbH = abH$.
>
> I first compute abH (because I see abH has a simpler definition that the other set).
>
> Computing abH, I get $abH = r^2H = \{r^2, r^2f\}$.
>
> Now, I try to find an element in $aHbH = rHrH$ which is not in r^2H. Since H has only two elements, to compute all elements of $aHbH$ I just need to compute $aebe$, $aebf$, $afbe$, and $afbf$. But I see that the first two are in abH by Definition of abH, so I will only check the last two elements.
>
> I try $afbe = rfr = f$, which is not in abH. This example would be enough to show that $rHrH \neq rrH$.
>
> (You can also try $a = b = rf$, or $a = r$ and $b = rf$, and see what happens.)

(c) In class, we proved that multiplication of cosets of N is well-defined if N is a normal subgroup.
Give an example where "multiplication" of cosets is not well-defined. That is, give a group G and a subgroup H where $a_1H = a_2H$ and $b_1H = b_2H$ but $a_1b_1H \neq a_2b_2H$.

> **Solution:** You can use the same G and H as in the previous question. Just make sure your a_1, a_2, b_1, b_2 are not in H.
>
> Another possible example is the following:
> Consider the symmetric group S_3 and let $J := \langle (1\ 2) \rangle$.
> Then the three left cosets of J are:

(a) $J = \{e, (1\ 2)\}$,

(b) $(132)J = (13)J = (1\ 3), (1\ 3\ 2)\}$, and

(c) $(1\ 2\ 3)J = (2\ 3)J = \{(2\ 3), (1\ 2\ 3)\}$.

Take $a_1 := (132)$, $a_2 := (13)$,
$b_1 := (123)$, and $b_2 := (23)$.

Then $a_1 b_1 J = (132)(123)J = eJ = J$, but $a_2 b_2 J = (13)(23)J = (123)J \neq J$.

6. (a) Given two groups A and B, what is the definition of the set $A \times B$? What is the binary operation on $A \times B$

Solution: See Chapter 8 External direct products.

(b) What is the identity element of $A \times B$?

Solution: $(1_A, 1_B)$, where 1_A is the identity element of A, and 1_B is the identity element of B.

(c) If $(a, b) \in A \times B$, what is the inverse $(a, b)^{-1}$ equal to?

Solution: (a^{-1}, b^{-1})

(d) Assume that neither of A and B is the trivial group. Prove that these four subgroups are normal in $A \times B$:

$$\{e_A\} \times \{e_B\}, \qquad A \times \{e_B\}, \qquad \{e_A\} \times B, \qquad A \times B$$

7. (a) True or false? The order of the group D_n is the same as the order of the group $\mathbb{Z}_2 \times \mathbb{Z}_n$.

Solution: True, the order is $2n$ for both.

(b) True or false? The group D_n is isomorphic to the group $\mathbb{Z}_2 \times \mathbb{Z}_n$.

Solution: False. If $n \geq 3$, the Dihedral group D_n is non-abelian, but $\mathbb{Z}_2 \times \mathbb{Z}_n$ is.

(c) True or false? The group $\mathbb{Z}_{14}$ is isomorphic to the group $\mathbb{Z}_2 \times \mathbb{Z}_7$.

Solution: True.
A possible proof: Note that $\mathbb{Z}_2 \times Z_7$ can be generated by the single element $(1, 1) \in \mathbb{Z}_2 \times \mathbb{Z}_7$ which has order 14, the least common multiple of 2 and 7. So $\mathbb{Z}_2 \times Z_7$ is a cyclic group of order 14.

(d) True or false? The group $\mathbb{Z}_{16}$ is isomorphic to the group $\mathbb{Z}_4 \times \mathbb{Z}_4$.

Solution: False. The group $\mathbb{Z}_{16}$ contains an element of order 16, that is, the number 1. Every element in the group $\mathbb{Z}_4 \times \mathbb{Z}_4$ has order 1, 2, or 4, so it cannot be generated by just one element; thus $\mathbb{Z}_4 \times \mathbb{Z}_4$ is not a cyclic group.

(e) Which direct product is isomorphic to $\mathbb{Z}_{12}$?

Solution: The direct product $\mathbb{Z}_4 \times \mathbb{Z}_3$ is isomorphic to $\mathbb{Z}_{12}$, since it can be generated by the element $(1, 1)$ which has order 12.

8. Let H be a subgroup of G.

   (a) What does the notation G/H mean?

   Solution: The set of all left cosets of H in G, that is, $\{xH \mid x \in G\}$.

   (b) When is G/H a group?

   Solution: When H is a normal subgroup of G.

   (c) If G/N is a quotient group, what is the binary operation of the quotient group G/N?

   Solution: $(aN)(bN) := abN$.

   (d) Consider the symmetric group S_3 and a subgroup $H := \langle(1\ 2)\rangle$. Is the set $S_3/\langle(1\ 2)\rangle$ a quotient group? Prove your answer. If it is a quotient group, what is it isomorphic to?

   Solution: No, $S_3/\langle(1\ 2)\rangle$ is not a quotient group because H is not normal in S_3.
   A possible proof: The left coset $(123)\langle(1\ 2)\rangle = \{(23), (123)\}$ and the right coset $\langle(1\ 2)\rangle(123) = \{(13), (123)\}$ are not equal.
   Another way to see that H is not normal is to recall that there are conjugates of (12) which are not in H, namely, (13) and (23).

   (e) Consider the symmetric group S_3 and a subgroup $J := \langle(1\ 2\ 3)\rangle$. Is S_3/J a quotient group? Prove your answer. If it is a quotient group, what is it isomorphic to?

   Solution: Yes, S_3/J is a quotient group because J is normal in S_3.

   A possible proof: Since the order of S_3 is 6 and the order of J is 3, there are two left cosets of J. Hence the left coset of J (which is not J itself) must be equal to the right coset of J (which is not equal to J itself).

   The quotient froup S_3/J is isomorphic to $\mathbb{Z}_2$ since there are two left cosets of J in S_3.

9. The following are all normal subgroups of D_4:

   (a) The trivial subgroup $\{e\}$,

   (b) the only normal subgroup of order 2, $\langle r^2\rangle$,

   (c) all the subgroups of order 4: $\langle r\rangle$, $\langle r^2, f\rangle$, $\langle r^2, rf\rangle$, and

   (d) D_4 itself.

   For each N above, what familiar group is D_4/N isomorphic to?

Solution: The only one that we have to compute carefully is $D_4/\langle r^2\rangle$. We know that the number of cosets in $D_4/\langle r^2\rangle$ is 4, but there are two groups of order 4 (up to isomorphism), so let's list the cosets in $D_4/\langle r^2\rangle$:
$\langle r^2\rangle$, $r\langle r^2\rangle$, $f\langle r^2\rangle$, and $rf\langle r^2\rangle$.

By inspection, we see that each element (each coset) in $D_4/\langle r^2\rangle$ has order 2, so this quotient group must be isomorphic to V_4, and not to $\mathbb{Z}_4$.

Final answer:
$D_4/\{e\} \cong D_4$
$D_4/\langle r^2\rangle \cong V_4$,
For each subgroup H of order 4, we have $D_4/H \cong \mathbb{Z}_2$, and
$D_4/D_4 \cong \{e\}$.

10. Let H be a subgroup of G, and consider the subset of G denoted by

$$\mathrm{Nor}_G(H) = \{g \in G : gH = Hg\} = \{g \in G : gHg^{-1} = H\}.$$

(a) Prove that $\mathrm{Nor}_G(H)$ is a subgroup.

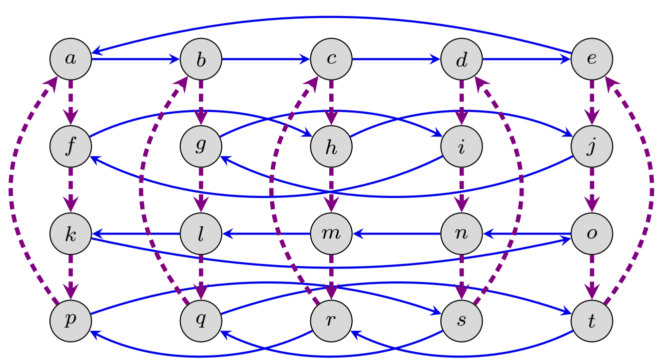
(b) What is the smallest that $\mathrm{Nor}_G(H)$ can be? What is the largest $\mathrm{Nor}_G(H)$ can be?

Solution: H and G (respectively)

(c) When does the latter happens?

Solution: $\mathrm{Nor}_G(H) = G$ if and only if H is normal.

11. Let G be the group whose Cayley diagram is shown below, and suppose e is the identity element. Consider the subgroups $A = \langle a\rangle = \{a, b, c, d, e\}$ and $J = \langle j\rangle = \{e, j, o, t\}$.



Carry out the following steps for both of the subgroups A and J. List the cosets element-wise.

(a) Write G as a disjoint union of the left cosets of A. Write G as a disjoint union of the left cosets of J.

(b) Write G as a disjoint union of the right cosets of A. Write G as a disjoint union of the right cosets of J.

(c) Use your coset computation to immediately compute the normalizer of the subgroup. Based on the computation for the normalizer, what you can say about this subgroup?

Solution:
$\mathrm{Nor}_G(A) = G$, which means $A \trianglelefteq G$.
$\mathrm{Nor}_G(J) = J$, which means that J is as "unnormal" as possible.

(d) Is G/A a group? If so, perform the quotient process and draw the resulting Cayley diagram for G/A.

> **Solution:** The quotient group G/A is isomorphic to $\mathbb{Z}_4$.

(e) Is G/J a group? If so, perform the quotient process and draw the resulting Cayley diagram for G/J.

> **Solution:** Since J is not normal, the set A/J is not a group.

12. The *center* of a group G is the set

$$Z(G) = \{z \in G \mid gz = zg, \text{ for all } g \in G\} = \{z \in G \mid gzg^{-1} = z, \text{ for all } g \in G\}.$$

It is a subgroup of G.

a. Prove that $Z(G)$ is normal in G by showing $ghg^{-1} \in H$ for all $h \in H, g \in G$ ("closed under conjugation").

> **Solution:** Suppose $g \in G$. By Theorem 1, it is sufficient to show that $gzg^{-1} \in Z(G)$ for all $z \in Z(G)$. But, if $z \in Z(G)$, then $gzg^{-1} = z \in Z(G)$ for all $g \in G$.

b. Compute the center of $\mathbb{Z}_6$. Compute the center of S_2.

> **Solution:** $\mathbb{Z}_6$ is abelian, so the entire group is the center. Similarly, S_2 is abelian (it's isomorphic to $\mathbb{Z}_2$) so the entire group is the center.

c. Compute the center of D_4.

> **Solution:** The center of D_4 is $\langle R^2 \rangle$. Reason: the half circle rotation commutes with every reflection (and every rotation). A different rotation does not commute with a reflection (for example, f). None of the reflections commutes with R.

d. Compute the center of D_5.

> **Solution:** The center of D_5 is the trivial group. Reason: None of the rotations commutes with f. None of the reflections commutes with R.

e. Consider the group A_3 of even permutations. Compute the center of A_3.

> **Solution:** A_3 is abelian, and therefore the center of A_3 is the entire group A_3.
> To see why A_3 is abelian, note that A_3 is a cyclic group of order 3, since it can be generated by the 3-cycle (123).
> Another way to see that A_3 is abelian, is to compute its order which is $3!/2 = 3$. We've seen that every group of order 3 (or any prime number) is cyclic.

f. Consider the group A_n of even permutations, where $n \geq 4$. Prove that (1 2 3) is not in the center of A_n by producing another even permutation which does not commute with (1 2 3).

> **Solution:** The element (2 3 4) works. (2 3 4)(1 2 3) = (12)(34)
> (1 2 3)(2 3 4) = (13)(24)

g. Let $n \geq 4$. Prove that $(1\ 2)(3\ 4)$ is not in the center of A_n.

> **Solution:** For example, you can show that the element $(1\ 2\ 3)$ does not commute with $(1\ 2)(3\ 4)$.

h. First, convince yourself that a non-identity permutation in S_4 is an even permutation if and only of its cycle notation is of the form $(ab)(cd)$ or (abc).
Compute the center of A_4 Hint: Do $(ab)(cd)$ and (abc) commute?

> **Solution:** Answer: The answer is the trivial group.
> Reason: The permutations (abc) and $(ab)(cd)$ do not commute.
> $(abc)(ab)(cd) = (a)(bdc)$ and $(ab)(cd)(abc) = (acd)(b)$.

i. Compute the center of S_4.
Hint: Every non-identity permutation in S_4 can be written in the form (ab), (abc), $(abcd)$, and $(ab)(cd)$. Can you find a permutation that does not commute with (ab)? With $(abcd)$?

j. Prove that "the center of a direct product is the direct product of the centers", that is, $Z(A \times B) = Z(A) \times Z(B)$.

> **Solution:** First, it is clear that $Z(A \times B) \supset Z(A) \times Z(B)$.
> To show that $Z(A \times B) \subset Z(A) \times Z(B)$, let $(z_1, z_2) \in Z(A \times B)$. Then, by definition, $(z_1, z_2)(g_1, g_2) = (g_1, g_2)(z_1, z_2)$ for all $g_1 \in A$, $g_2 \in B$. This means that $(z_1 g_1, z_2 g_2) = (g_1 z_1, g_2 z_2)$ for all $g_1 \in A$, $g_2 \in B$. In other words, $z_1 g_1 = g_1 z_1$ and $z_2 g_2 = g_2 z_2$ for all $g_1 \in A$, $g_2 \in B$, so $z_1 \in Z(A)$ and $z_2 \in Z(B)$.

13. Notation/Definition: Let G be a group and $x \in G$.

- The *conjugacy* class of x is the set $\mathrm{cl}_G(x) := \{gxg^{-1} \mid g \in G\}$.
- Let $Z(G)$ be the set $\{z \in G \mid gz = zg \text{ for all } g \in G\}$.

Suppose N is a normal subgroup of G. Prove that if $x \in N$, then $\mathrm{cl}_G(x) \subset N$.

> **Solution:** Let $x \in N$. Since N is normal in G, we have $gxg^{-1} \in N$ for all $g \in G$. So $\mathrm{cl}_G(x) := \{gxg^{-1} : g \in G\} \subset N$.

14. You can use the following fact.

Proposition 1. For any $\sigma \in S_n$, we have $\sigma\ (a_1\ a_2\ \dots\ a_k)\ \sigma^{-1} = (\,\sigma(a_1)\ \sigma(a_2)\ \dots\ \sigma(a_k)\,)$.

(a) Let x be a k-cycle. Prove that $y \in S_n$ is conjugate to x iff y is a k-cycle.

> **Solution:** By Proposition 1, every pair of k-cycles are conjugate.

(b) Prove that (12) and (14) in S_6 are conjugate by finding a permutation $p \in S_6$ such that $p^{-1}(12)p = (14)$.

(c) List all permutations in S_4 which are conjugate to (1234). Use the fact from part (a).

> **Solution:** The answer is $(1234), (1432), (1243), (1342), (1324), (1423)$. Explanation: The permutations which are conjugate to (1234) in S_4 are all the 4-cycles.