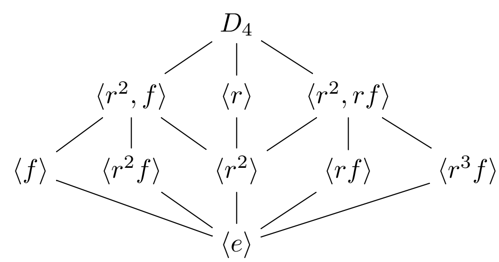
1. (a) Let $n > 1$. Let A_n and B_n denote the set of even permutations and the set of odd permutations, respectively. Define a map $f : A_n \to B_n$ by $f(\pi) = (1\,2)\pi$ for all $\pi \in A_n$.
   Prove that this map is injective and surjective.
   (b) Let H be a subgroup of a group G, and let $x \in G$. Define a bijective map f from H to xH.
   (c) Show that this map is surjective.
   (d) Suppose G is a non-abelian group of order 1000 and H is a subgroup of order 20. Let x be an element of G which is not in H.
   (i) How many elements are in the left coset xH?
   (ii) How many elements are in the right coset Hx?
   (iii) How many left cosets of H are there?

2. (a) I listed all subgroups of D_4 (in a subgroup lattice) below. Label each edge between $K \leq H$ with the index $[H : K]$.



   (b) Is $f\langle r\rangle = \langle r\rangle f$? What about other left and right cosets of $\langle r\rangle$? Prove your answer.
   (c) Is the left coset $r^3f\langle r^2, f\rangle$ equal to the right coset $\langle r^2, f\rangle r^3 f$?

3. (a) If H is a subgroup of G and $a \in G$, then a left coset aH is ... [give the definition]
   (b) The *index* $[G : H]$ of a subgroup $H \leq G$ is [give a definition, not a theorem!] ...

Theorem 1. Let H be a subgroup of G. Then the following are all equivalent.

(i) The subgroup H is called *normal* in G, that is, $gH = Hg$ for all $g \in G$; ("left cosets are right cosets");
(ii) $ghg^{-1} \in H$ for all $h \in H, g \in G$; ("closed under conjugation").
(iii) $gHg^{-1} = H$ for all $g \in G$; ("only one conjugate subgroup")

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4. (a) Consider the subgroup $H = \{(1), (1, 2)\}$ of S_3. Is H normal?
   (b) Consider the subgroup $J = \{(1), (123), (132)\}$ of S_3. Is J normal?
   (c) Consider the subgroup $H = \langle (1234)\rangle$ of S_4. Is H normal?
   (d) Let $n > 2$. Is A_n a normal subgroup of S_n?
   (e) Consider a mystery subgroup K of $\mathbb{Z}_5 \times \mathbb{Z}_8$. Is K normal?

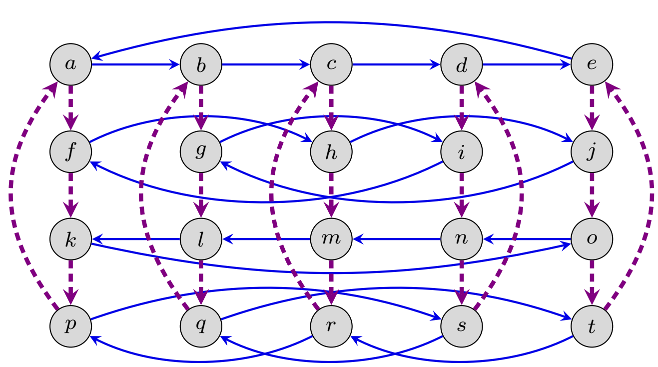
5. Let H be a subgroup of G. Given two fixed elements $a, b \in G$, define the sets

$$aHbH := \{ah_1bh_2 : h_1, h_2 \in H\} \qquad \text{and} \qquad abH := \{abh : h \in H\}.$$

   (a) Prove that *if* H is normal then $aHbH \subset abH$.
   (b) Prove that the statement is false if we remove the "normal" assumption. That is, give a specific G and H and $a, b \in G$ such that $aHbH$ is not a subset of abH.
   (c) In class, we proved that multiplication of cosets of N is well-defined if N is a normal subgroup.
   Give an example where "multiplication" of cosets is not well-defined. That is, give a group G and a subgroup H where $a_1H = a_2H$ and $b_1H = b_2H$ but $a_1b_1H \neq a_2b_2H$.

6. (a) Given two groups A and B, what is the definition of the set $A \times B$? What is the binary operation on $A \times B$
   (b) What is the identity element of $A \times B$?
   (c) If $(a, b) \in A \times B$, what is the inverse $(a, b)^{-1}$ equal to?
   (d) Assume that neither of A and B is the trivial group. Prove that these four subgroups are normal in $A \times B$:
$$\{e_A\} \times \{e_B\}, \qquad A \times \{e_B\}, \qquad \{e_A\} \times B, \qquad A \times B$$
7. (a) True or false? The order of the group D_n is the same as the order of the group $\mathbb{Z}_2 \times \mathbb{Z}_n$.
   (b) True or false? The group D_n is isomorphic to the group $\mathbb{Z}_2 \times \mathbb{Z}_n$.
   (c) True or false? The group $\mathbb{Z}_{14}$ is isomorphic to the group $\mathbb{Z}_2 \times \mathbb{Z}_7$.
   (d) True or false? The group $\mathbb{Z}_{16}$ is isomorphic to the group $\mathbb{Z}_4 \times \mathbb{Z}_4$.
   (e) Which direct product is isomorphic to $\mathbb{Z}_{12}$?
8. Let H be a subgroup of G.
   (a) What does the notation G/H mean?
   (b) When is G/H a group?
   (c) If G/N is a quotient group, what is the binary operation of the quotient group G/N?
   (d) Consider the symmetric group S_3 and a subgroup $H := \langle (1\ 2) \rangle$. Is the set $S_3 / \langle (1\ 2) \rangle$ a quotient group? Prove your answer. If it is a quotient group, what is it isomorphic to?
   (e) Consider the symmetric group S_3 and a subgroup $J := \langle (1\ 2\ 3) \rangle$. Is S_3/J a quotient group? Prove your answer. If it is a quotient group, what is it isomorphic to?
9. The following are all normal subgroups of D_4:
   (a) The trivial subgroup $\{e\}$,
   (b) the only normal subgroup of order 2, $\langle r^2 \rangle$,
   (c) all the subgroups of order 4: $\langle r \rangle$, $\langle r^2, f \rangle$, $\langle r^2, rf \rangle$, and
   (d) D_4 itself.

   For each N above, what familiar group is D_4/N isomorphic to?
10. Let H be a subgroup of G, and consider the subset of G denoted by
$$\mathrm{Nor}_G(H) = \{g \in G : gH = Hg\} = \{g \in G : gHg^{-1} = H\}.$$
   (a) Prove that $\mathrm{Nor}_G(H)$ is a subgroup.
   (b) What is the smallest that $\mathrm{Nor}_G(H)$ can be? What is the largest $\mathrm{Nor}_G(H)$ can be?
   (c) When does the latter happens?
11. Let G be the group whose Cayley diagram is shown below, and suppose e is the identity element. Consider the subgroups $A = \langle a \rangle = \{a, b, c, d, e\}$ and $J = \langle j \rangle = \{e, j, o, t\}$.



Carry out the following steps for both of the subgroups A and J. List the cosets element-wise.

(a) Write G as a disjoint union of the left cosets of A. Write G as a disjoint union of the left cosets of J.

(b) Write G as a disjoint union of the right cosets of A. Write G as a disjoint union of the right cosets of J.

(c) Use your coset computation to immediately compute the normalizer of the subgroup. Based on the computation for the normalizer, what you can say about this subgroup?

(d) Is G/A a group? If so, perform the quotient process and draw the resulting Cayley diagram for G/A.

(e) Is G/J a group? If so, perform the quotient process and draw the resulting Cayley diagram for G/J.

12. The *center* of a group G is the set

$$Z(G) = \{z \in G \mid gz = zg, \text{ for all } g \in G\} = \{z \in G \mid gzg^{-1} = z, \text{ for all } g \in G\}.$$

It is a subgroup of G.

a. Prove that $Z(G)$ is normal in G by showing $ghg^{-1} \in H$ for all $h \in H, g \in G$ ("closed under conjugation").

b. Compute the center of $\mathbb{Z}_6$. Compute the center of S_2.

c. Compute the center of D_4.

d. Compute the center of D_5.

e. Consider the group A_3 of even permutations. Compute the center of A_3.

f. Consider the group A_n of even permutations, where $n \geq 4$. Prove that (1 2 3) is not in the center of A_n by producing another even permutation which does not commute with (1 2 3).

g. Let $n \geq 4$. Prove that (1 2)(3 4) is not in the center of A_n.

h. First, convince yourself that a non-identity permutation in S_4 is an even permutation if and only of its cycle notation is of the form $(ab)(cd)$ or (abc).

Compute the center of A_4 Hint: Do $(ab)(cd)$ and (abc) commute?

i. Compute the center of S_4.

Hint: Every non-identity permutation in S_4 can be written in the form (ab), (abc), $(abcd)$, and $(ab)(cd)$. Can you find a permutation that does not commute with (ab)? With $(abcd)$?

j. Prove that "the center of a direct product is the direct product of the centers", that is, $Z(A \times B) = Z(A) \times Z(B)$.

13. Notation/Definition: Let G be a group and $x \in G$.

- The *conjugacy* class of x is the set $\text{cl}_G(x) := \{gxg^{-1} \mid g \in G\}$.
- Let $Z(G)$ be the set $\{z \in G \mid gz = zg \text{ for all } g \in G\}$.

Suppose N is a normal subgroup of G. Prove that if $x \in N$, then $\text{cl}_G(x) \subset N$.

14. You can use the following fact.

Proposition 1. For any $\sigma \in S_n$, we have $\sigma\ (a_1\ a_2\ \ldots\ a_k)\ \sigma^{-1} = (\,\sigma(a_1)\ \sigma(a_2)\ \ldots\ \sigma(a_k)\,)$.

(a) Let x be a k-cycle. Prove that $y \in S_n$ is conjugate to x iff y is a k-cycle.

(b) Prove that (12) and (14) in S_6 are conjugate by finding a permutation $p \in S_6$ such that $p^{-1}(12)p = (14)$.

(c) List all permutations in S_4 which are conjugate to (1234). Use the fact from part (a).