Gallian's textbook 9th ed "Contemporary abstract algebra" Chapter 14 Example 12

$$\langle x^2 + 1\rangle = \{f(x)(x^2 + 1) \mid f(x) \in \mathbf{R}[x]\}.$$

Then

$$\begin{aligned}\mathbf{R}[x]/\langle x^2 + 1\rangle &= \{g(x) + \langle x^2 + 1\rangle \mid g(x) \in \mathbf{R}[x]\} \\ &= \{ax + b + \langle x^2 + 1\rangle \mid a, b \in \mathbf{R}\}.\end{aligned}$$

To see this last equality, note that if $g(x)$ is any member of $\mathbf{R}[x]$, then we may write $g(x)$ in the form $q(x)(x^2 + 1) + r(x)$, where $q(x)$ is the quotient and $r(x)$ is the remainder upon dividing $g(x)$ by $x^2 + 1$. In particular, $r(x) = 0$ or the degree of $r(x)$ is less than 2, so that $r(x) = ax + b$ for some a and b in $\mathbf{R}$. Thus,

$$\begin{aligned}g(x) + \langle x^2 + 1\rangle &= q(x)(x^2 + 1) + r(x) + \langle x^2 + 1\rangle \\ &= r(x) + \langle x^2 + 1\rangle,\end{aligned}$$

since the ideal $\langle x^2 + 1\rangle$ absorbs the term $q(x)(x^2 + 1)$.

How is multiplication done? Since

$$x^2 + 1 + \langle x^2 + 1\rangle = 0 + \langle x^2 + 1\rangle,$$

one should think of $x^2 + 1$ as 0 or, equivalently, as $x^2 = -1$. So, for example,

$$\begin{aligned}&(x + 3 + \langle x^2 + 1\rangle) \cdot (2x + 5 + \langle x^2 + 1\rangle) \\ &\quad = 2x^2 + 11x + 15 + \langle x^2 + 1\rangle = 11x + 13 + \langle x^2 + 1\rangle.\end{aligned}$$

In view of the fact that the elements of this ring have the form $ax + b + \langle x^2 + 1\rangle$, where $x^2 + \langle x^2 + 1\rangle = -1 + \langle x^2 + 1\rangle$, it is perhaps not surprising that this ring turns out to be algebraically the same ring as the ring of complex numbers. This observation was first made by Cauchy in 1847. ∎

Examples 11 and 12 illustrate one of the most important applications of factor rings—the construction of rings with highly desirable properties. In particular, we shall show how one may use factor rings to construct integral domains and fields.

Prime Ideals and Maximal Ideals

Definition Prime Ideal, Maximal Ideal

A *prime ideal* A of a commutative ring R is a proper ideal of R such that $a, b \in R$ and $ab \in A$ imply $a \in A$ or $b \in A$. A *maximal* ideal of a commutative ring R is a *proper* ideal of R such that, whenever B is an ideal of R and $A \subseteq B \subseteq R$, then $B = A$ or $B = R$.