1 Question

Demonstrate that the ideal $9\mathbb{Z}$ is not a maximal ideal of $\mathbb{Z}$ by providing a different ideal J of $\mathbb{Z}$ which contains $9\mathbb{Z}$.

Solution: The ideal $3\mathbb{Z}$ properly contains $9\mathbb{Z}$.

2 Question

The quotient ring $\mathbb{R}[x]/\langle x-10\rangle$ is isomorphic to $\mathbb{R}$ by applying the 1st isomorphism theorem using the evaluation homomorphism

$$\begin{aligned} \varphi : \mathbb{R}[x] &\to \mathbb{R} \quad \text{defined by} \\ p(x) &\mapsto p(10). \end{aligned}$$

Is the principal ideal $\langle x-10\rangle$ a maximal ideal in $\mathbb{R}[x]$?

Solution: Since $\mathbb{R}$ is a field, the quotient ring $\mathbb{R}[x]/\langle x-10\rangle$ is a field. Because of Theorem 16.35, the ideal $\langle x-10\rangle$ is maximal in $\mathbb{R}[x]$.

3 Question

Demonstrate that $9\mathbb{Z}$ is not a prime ideal of $\mathbb{Z}$ (by finding elements a, b such that $ab \in 9\mathbb{Z}$ but $a \notin 9\mathbb{Z}$ and $b \notin 9\mathbb{Z}$).

Solution: $a = 3$, $b = 3$ work.
$a = 21$, $b = 24$ also work.
$a = 30$, $b = 600$ also work.

4 Question

Let R be a commutative ring with unity. Prove: if I is a maximal ideal of R then I is also a prime ideal of R.

Solution: Suppose I is a maximal idea of R. A theorem from Section 16.4 tells us R/I is a field. So R/I is an integral domain by definition. Another theorem from Section 16.4 tells us that I is prime ideal.

If you finish the quiz early, work on the following problem. (This page will not be graded.)

Let G be a group. Consider the map

$$\begin{aligned} \phi : G &\to \mathrm{Perm}(G) \quad \text{defined by} \\ g &\mapsto T_g \end{aligned}$$

where T_g is the bijection from G to itself (that is, T_g is a permutation on G) defined by

$$T_g(x) = gx \quad \text{for all } x \in G$$

(i.e., T_g is multiplication by g on the left).

(a) Prove that ϕ is a group homomorphism.

(b) Prove that ϕ is injective.

(c)
- Pick a familiar group G with small order (say, 3 or 4 or 6). Write the Cayley table (group operation table) for this G. Then use it to compute the permutation T_g for each g in your chosen group G.

- Rename the elements g of your chosen group G so that the elements of G are $1, 2, \ldots, |G|$. Rewrite each T_g in cycle notation. Draw the Cayley graph of $\{T_g : g \in G\}$ with vertices written in cycle notation.

Note: This homomorphism ϕ is called the *left regular representation* of G.