First & Last Name: ______________________ Student ID: ______________

Let $\mathbb{R}[x]$ denote the ring of all polynomials whose coefficients are real numbers.

1. Consider the principal ideal $I = \langle\ (x-3)\ \rangle = \{f(x)(x-3) : f(x) \in \mathbb{R}[x]\}$ of $\mathbb{R}[x]$.

   (a) This ideal I has infinitely many elements. Write three of the elements.

   Solution: $(x-3)$, 0, and $x^4(x-3)$

   (b) There are infinitely many elements (cosets) in the quotient ring $\mathbb{R}[x]/I$. Write three distinct elements (cosets) in $\mathbb{R}[x]/I$.

   Solution: I, $1+I$, and x^3+I

   (c) Consider the coset $(x^2+x+5)+I$ and the coset $(x+5)+I$. Are they the same element in the quotient ring $\mathbb{R}[x]/I$? (Circle **Yes**/ **no**)

   Solution: No.
   $(x^2+x+5)-(x+5) = x^2 \notin I$

   (d) Are the cosets $(x^2+x+5)+I$ and $(4x+5)+I$ the same set? (Circle **Yes**/ **no**)

   Solution: Yes.
   $(x^2+x+5)-(4x+5) = x^2-3x \in I$

   (e) What is the zero element in the quotient ring $\mathbb{R}[x]/I$?

   Solution: The ideal I is the zero element (coset) in the quotient ring.

   (f) The quotient ring $\mathbb{R}[x]/I$ has unity. What is the unity element (coset)?

   Solution: $1+I$ is the unity element (coset) in the quotient ring.

2. Consider the evaluation homomorphism

$$\varphi : \mathbb{R}[x] \to \mathbb{R} \quad \text{defined by}$$
$$p(x) \mapsto p(3)$$

   For example, $\varphi(x^4 - \frac{1}{2}) = 3^4 - \frac{1}{2} = 81 - \frac{1}{2} = 80\frac{1}{2}$.

   (a) The kernel of φ is equal to the ideal I defined above, and the image of φ is the ring (field) of real numbers $\mathbb{R}$ (You don't need to prove these). Using these facts, apply the 1st isomorphism theorem (for rings) to this situation.

   Solution: The 1st isomorphism theorem says that the quotient ring $\mathbb{R}[x]/\ker\varphi$ is isomorphic to $\text{im}\varphi$, so the two rings $\mathbb{R}[x]/\langle\ (x-3)\ \rangle$ and $\mathbb{R}$ are isomorphic.

(b) The First isomorphism theorem gives us the injective ring homomorphism

$$\iota : \mathbb{R}[x]/\langle(x-3)\rangle \to \mathbb{R}$$
$$p(x) + \langle(x-3)\rangle \mapsto \varphi(p(x))$$

For these last three questions, your answers should be specific numbers (simplified).

(i) Where does ι send the coset $(x^2 + x + 5) + I$?

Solution: $\iota(\ (x^2+x+5)+I\ ) = \varphi(x^2+x+5) = 3^2+3+5 = \boxed{17}$

(ii) Where does ι send the coset $(x+5)+I$?

Solution: $\iota(\ (x+5)+I\ ) = \varphi(x+5) = 3+5 = \boxed{8}$

(iii) Where does ι send the coset $(4x+5)+I$?

Solution: $\iota(\ (4x+5)+I\ ) = \varphi(4x+5) = 4(3)+5 = \boxed{17}$

If you finish the quiz early, work on the following problem. It will be discussed during class. (This page will not be graded.)

From real polynomials to complex numbers

Consider the evaluation homomorphism

$$\varphi : \mathbb{R}[x] \to \mathbb{C} \quad \text{defined by}$$
$$p(x) \mapsto p(i)$$

(a) What is the kernel of φ?

> **Solution:** The set of real polynomials which have i as a root.
> This set is equal to the set of real polynomials having $(x^2 + 1)$ as a factor,
> $$\{f(x)(x^2 + 1) : f(x) \in \mathbb{R}[x]\}$$

(b) What is the image of φ?

> **Solution:** The ring (field) of complex numbers $\mathbb{C}$.

(c) Apply the 1st isomorphism theorem (for rings) to this specific situation.

> **Solution:** The quotient ring $\mathbb{R}[x]/\langle\ x^2 + 1\ \rangle$ is isomorphic to $\mathbb{C}$.

(d) Since $\mathbb{R}[x]$ is a commutative ring, the quotient ring $\mathbb{R}[x]/(\ker \varphi)$ is also commutative. Since the constant polynomial 1 is not in $\ker \varphi$, the element (coset)

$$1 + \ker \varphi$$

is not equal to the zero element (coset)

$$\ker \varphi,$$

so $\mathbb{R}[x]/(\ker \varphi)$ is a commutative ring with unity $1 + \ker \varphi$.

What type of ring is $\mathbb{R}[x]/(\ker \varphi)$? Does it have zero divisors? Is it an integral domain? Is it a field?

> **Solution:** Since $\mathbb{C}$ is a field and the quotient ring $\mathbb{R}[x]/(\ker \varphi)$ is isomorphic to $\mathbb{C}$, the quotient ring $\mathbb{R}[x]/(\ker \varphi)$ is a field.