1 Kernel

Let $\varphi : S \to T$ be a ring homomorphism.

(a) Write the definition of the kernel of φ.

> **Solution:**
> $$\{x \in S : \varphi(x) = \mathbf{0}_T\}$$

(b) Let $K = \ker \varphi$. Prove that $xK \subset K$ for all $x \in S$, that is, prove that,

for all $x \in S$ and $a \in K$, the product xa is in K.

> **Solution:** Suppose $x \in S$ and $a \in K$. Then
>
> $$\begin{aligned} \varphi(xa) &= \varphi(x)\ \varphi(a) \text{ because } \varphi \text{ being a ring homomorphism means that it preserves the ring multiplication} \\ &= \varphi(x)\ \mathbf{0}_T \text{ since } a \in K = \ker \varphi \\ &= \mathbf{0}_T \text{ by definition of the zero element} \end{aligned}$$
>
> So $xa \in \ker \varphi$.

2 Evaluation honomorphism

Let $\mathbb{R}[x]$ denote the ring of all polynomials whose coefficients are real numbers. Consider the evaluation homomorphism

$$\begin{aligned} \varphi : \mathbb{R}[x] &\to \mathbb{R} \quad \text{defined by} \\ p(x) &\mapsto p(3) \end{aligned}$$

(a) What is the kernel of φ?

> **Solution:** The set of real polynomials which have 3 as a root.
>
> Since $\mathbb{R}$ is a field, this set is equal to the set of real polynomials having $(x-3)$ as a factor,
> $$\{f(x)(x-3) : f(x) \in \mathbb{R}[x]\}$$

(b) What is the image of φ?

Solution: The ring (field) of real numbers $\mathbb{R}$.

3 Matrices with integer entries

Consider the ring $\mathrm{Mat}_2(\mathbb{Z}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) : a, b, c, d \in \mathbb{Z} \right\}$ of 2×2 matrices with integer entries, under the usual matrix addition and multiplication. Let I be the subset of $\mathrm{Mat}_2(\mathbb{Z})$ consisting of matrices with even entries.

(a) Prove one of the "absorbing" properties of I:

For all $X \in \mathrm{Mat}_2(\mathbb{Z})$ and $A \in I$, we have $XA \in I$.

Solution: Suppose $X = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{Mat}_2(\mathbb{Z})$ and $A = \left(\begin{smallmatrix} 2e & 2f \\ 2g & 2h \end{smallmatrix} \right) \in I$. Then

$$\begin{aligned} XA &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2e & 2f \\ 2g & 2h \end{pmatrix} \\ &= \begin{pmatrix} 2ae + 2bg & 2af + 2bh \\ 2ce + 2dg & 2cf + 2dh \end{pmatrix} \end{aligned}$$

is in I since XA has even entries.

(b) Is I an ideal of $\mathrm{Mat}_2(\mathbb{Z})$? Simply circle **Yes**/ **No**.

Solution: Yes

If you finish the quiz early, work on the following problem. It will be discussed during class. (This page will not be graded.)

Recall the following lemma about cosets.

Lemma 1. If H is an additive subgroup of the group $(G, +)$, the following are equivalent:

(1) $a + H = b + H$

(2) $b \in a + H$

(3) $b - a \in H$

4 Multiplication of cosets is well-defined

Let R be a ring, let $s, s', t, t' \in R$, and let I be an ideal of R. Suppose also that

$$s' \in s + I \text{ and } t' \in t + I.$$

Show that

$$s't' \in st + I \tag{4.1}$$

Solution: Since $s' \in s + I$, we have $s' = s + a$ for some $a \in I$. Similarly, since $t' \in t + I$, we have $t' = t + b$ for some $b \in I$. Then

$$s't' = (s + a)(t + b) = st + (sb + at + ab)$$

We have $sb, at, ab \in I$ because I is an ideal (and therefore satisfies the "absorbing" property). So $s't' \in st + I$.

(Note that (4.1) implies $s't' + I = st + I$, due to Lemma 1)