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1 Units

Definition 1. Let R be a ring which has unity denoted by the symbol $\mathbf{1}$. An element $u \in R$ is called a *unit* (also called an *invertible element*) if there exists $v \in R$ such that $uv = vu = \mathbf{1}$.

Suppose R is a ring with unity $\mathbf{1}$ and $x \in R$. Prove the following:

$$\text{if } x^2 = \mathbf{0} \text{ then } \mathbf{1} - x \text{ is a unit.}$$

Solution: Let $v = \mathbf{1} + x$. Then we have

$$\begin{aligned}(\mathbf{1} - x)v &= (\mathbf{1} - x)(\mathbf{1} + x) \\ &= \mathbf{1} + x - x(\mathbf{1} + x) \\ &= \mathbf{1} - x^2 \\ &= \mathbf{1} \text{ since } x^2 = \mathbf{0}\end{aligned}$$

2 Characteristic

Definition 2. We define the *characteristic* of a ring R to be the least positive integer such that

$$n \cdot r = \underbrace{r + r + ... + r}_{n \text{ copies}} = \mathbf{0}$$

for all $r \in R$. If no such integer exists, then the characteristic of R is defined to be 0.

(a) What is the characteristic of the ring $\mathbb{Z}_{40}$?

Solution: The order of the unity element 1 is 40, so, by Lemma 16.18 in Textbook section 16.2, the characteristic of the ring $\mathbb{Z}_{40}$ is 40.

(b) What is the characteristic of the ring $\mathbb{Z}$ of integers?

Solution: There is no positive integer n such that $n \cdot 1 = 0$, so the characteristic of this ring is 0.

3 Zero divisors

Definition 3. A nonzero element d in a commutative ring R is called a *zero divisor* if there is a nonzero element c in R such that $dc = \mathbf{0}$.

(a) What are the zero divisors (if any) of the ring $\mathbb{Z}_8$?

> **Solution:** All the nonzero elements which are not units: $2, 4, 6$. For example, 6 is a zero divisor because $(6)(4) = 0$.

(b) What are the zero divisors (if any) of the ring $\mathbb{Z}$ of integers?

> **Solution:** There are no zero divisors

4 Rings

(a) Let S be the set of 2×2 matrices with even integers as entries. In set-builder notation,

$$S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in 2\mathbb{Z} \right\}$$

Prove that S is closed under taking additive inverse (where addition is the usual matrix addition).

> **Solution:** Suppose $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S$. Then its additive inverse is $\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$ since their sum is the zero matrix.
> We can conclude that $\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} \in S$ because $-z$ is even for all even integers z.

(b) Let T be the set of 2×2 matrices of the form

$$\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$$

where $x, y \in \mathbb{Z}$. In set-builder notation, $T = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} : x, y \in \mathbb{Z} \right\}$.

Prove that T is closed under the usual matrix multiplication.

> **Solution:** Suppose $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} \in T$. Then $a, b, c, d \in \mathbb{Z}$, and so $ac, ad \in \mathbb{Z}$. Then
>
> $$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} ac & ad \\ 0 & 0 \end{pmatrix} \in T$$
>
> since $ac, ad \in \mathbb{Z}$.