First & Last Name: ______________________ Student ID: ______________

Let $\mathbb{Z}$ denote the additive group of integers and let $\mathbb{C}^*$ denote the multiplicative group of nonzero complex numbers, as usual.

1. Consider the homomorphism $f : \mathbb{Z} \to \mathbb{C}^*$ defined by

$$f(m) = i^m$$

(a) What is the kernel of f?

Solution: $4\mathbb{Z} = \{4m : m \in \mathbb{Z}\} = \{..., -8, -4, 0, 4, 8, ...\}$

(b) What is the image of f?

Solution: $\{1, i, -1, -i\} = \{1, e^{i(\pi/2)}, e^{i(\pi/2)2}, e^{i(\pi/2)3}\}$, the subgroup $\mathbb{T}_4$ of $\mathbb{C}^*$ consisting of the 4-th roots of unity.

(c) Apply the First Isomorphism Theorem to this situation. What does the theorem tell us about f?

Solution: We have

$$\mathbb{Z}/4\mathbb{Z} \cong \mathbb{T}_4$$

because the image of f is equal to $\mathbb{T}_4$

2. (a) True or false? The group $\mathbb{Z}_{40}$ is isomorphic to the group $\mathbb{Z}_8 \times \mathbb{Z}_5$.

Circle **True/ False**

(b) Give a brief reasoning for your selection above.

Solution: True because 8 and 5 are relatively prime.
Alternatively, you can use $(1, 1) \in \mathbb{Z}_8 \times \mathbb{Z}_5$ to generate the entire group, which shows that it is a cyclic group of order $8 \cdot 5 = 40$.

3. List all isomorphism classes of *abelian* groups of order $40 = 2^3 \cdot 5$.

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Solution:

- $\mathbb{Z}_8 \times \mathbb{Z}_5$ and other groups isomorphic to it
- $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_5$ and other groups isomorphic to it
- $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$ and other groups isomorphic to it