First & Last Name: ______________________________ Student ID: ____________________

1. Consider the group $\langle \sigma \rangle$, where

$$\sigma = (1263) = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 1 & 4 & 5 & 3 \end{bmatrix} \in S_6$$

   (a) If the group is finite, list all elements; if the group is infinite, describe all elements.

   Solution:

$$\{(1263), (16)(23), (3621), \text{Id}\} = \{\sigma, \sigma^2, \sigma^3, \text{Id}\}$$

   (b) Which group is the same as $\langle \sigma \rangle$? (fill in the correct square)

   ☐ $\mathbb{Z}_2$ ☐ $\mathbb{Z}_3$ ☐ $\mathbb{Z}_4$ ☐ $\mathbb{Z}_6$ ☐ $\mathbb{Z}$ ☐ $\mathbb{Z}_2 \times \mathbb{Z}_2$

   Solution: The group is a cyclic group of order 4, so the correct choice is $\mathbb{Z}_4$.

2. Consider the group $\langle R \rangle$, where R is the counterclockwise rotation by 90^o.

   (a) If the group is finite, list all elements; if the group is infinite, describe all elements.

   Solution:

$$\{\text{rotation by } 90^o, \text{rotation by } 180^o, \text{rotation by } 270^o, \text{Id}\} = \{R, R^2, R^3, \text{Id}\}$$

   (b) Which group is the same as $\langle R \rangle$? (fill in the correct square)

   ☐ $\mathbb{Z}_2$ ☐ $\mathbb{Z}_3$ ☐ $\mathbb{Z}_4$ ☐ $\mathbb{Z}_6$ ☐ $\mathbb{Z}$ ☐ $\mathbb{Z}_2 \times \mathbb{Z}_2$

   Solution: The group is a also cyclic group of order 4, so the correct choice is $\mathbb{Z}_4$.

3. Suppose w, x, y, z are elements of a group G, and suppose $|x| = 6$ and $|z| = 5$. Express

$$w^3 \; (x^4 \; y^{-8} \; z^2)^{-1}$$

   without using negative exponents.

   Explain each step.

   Solution:

$$\begin{aligned} w^3(x^4y^{-8}z^2)^{-1} &= w^3z^{-2}y^8x^{-4} \text{ by the "socks-shoes" property} \\ &= \boxed{w^3z^3y^8x^2} \text{ because } z^2z^3 = z^5 = e \text{ and } x^4x^2 = x^6 = e \end{aligned}$$

If you finish the quiz early, work on the following problem. The following group has not yet appeared in class but will come up in the future. (This page will not be graded)

Let

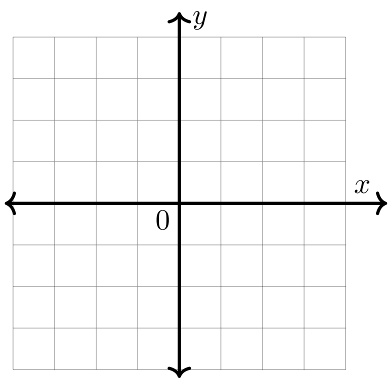
$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

where $i^2 = -1$.

The set $Q_8 = \{e, -e, I, -I, J, -J, K, -K\}$ together with matrix multiplication forms a group whose identity element is the matrix e.

(a) What does the matrix $I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ do to a vector $\begin{pmatrix} a \\ b \end{pmatrix}$ in $\mathbb{R}^2$?

$\Big($Compute $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$ for different values of $a, b \in \mathbb{R}$ and plot the vectors below. Hint: The matrix I has order 4, so it cannot be a reflection, since a reflection is its own inverse.$\Big)$



(b) We can check that

$$IJ = K,\ JK = I,\ KI = J,$$
$$IJ = -K,\ KJ = -I,\ KI = -J,$$
$$I^2 = -e,\ J^2 = -e,\ K^2 = -e.$$

Use these equalities to help you compute the order of each of the eight elements of Q_8.

Solution: The identity e has order 1; $-e$ has order 2. All other six elements have order 4:

The order of e is 1.

The order of $-e$ is 2 since $(-e)(-e) = e$.

$I^2 = -e \neq e$, so $I^3 = -I \neq e$, and $I^4 = I^2 I^2 = (-e)(-e) = e$. So the order of I is 4. $(-I)^2 = (-1)^2(I^2) = I^2 = -e \neq e$, so $(-I)^3 = I \neq e$, and $(-I)^4 = e$. So the order of $-I$ is 4.

By similar computation, the order of each of J, K, $-J$, and $-K$ is also 4.

(c) Explain why Q_8 is distinct from $\mathbb{Z}_8$ and also distinct from D_4.

Solution: A possible explanation is this Q_8 has 6 elements of order four. D_4 only has 2 such elements (two of the rotations), and $\mathbb{Z}_8$ only has 2 such elements (2 and 6).