First & Last Name: ________________________ Student ID: ______________

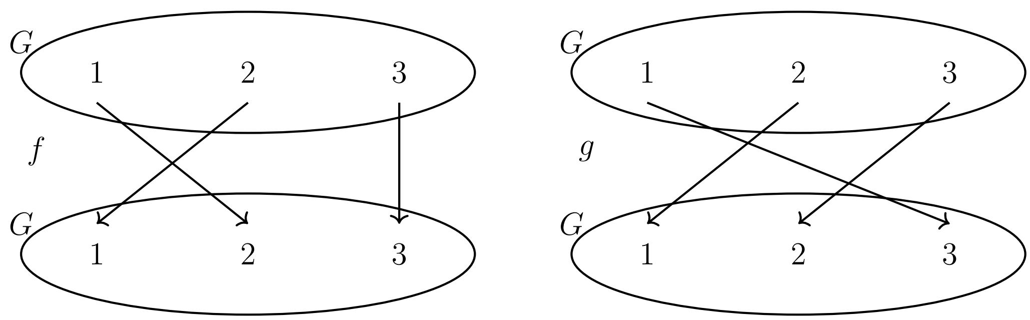
Abstract Algebra Individual Quiz 1

$\boxed{\textbf{1.}}$ Consider the group G of all bijections $\{1, 2, 3\} \to \{1, 2, 3\}$ together with function composition $\circ$ as binary operation.

Show that $(G, \circ)$ is not abelian.

Solution: We need to show that the group operation is not commutative by finding two elements $f, g \in G$ such that $f \circ g \neq g \circ f$.

Consider the bijections f, g below:



We can use the two-row notation to denote f and g:

$$f = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix} \text{ and } g = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

We have

$$f \circ g = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \text{ and } g \circ f = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix},$$

so $f \circ g \neq g \circ f$

2. Let $\star$ be an associative binary operation on a set S with identity e.

Prove the following: If $x \star x = e$ for all elements $x \in S$, then $\star$ is commutative.

Solution: Suppose $x, y \in S$. Then $xy \in S$. So, by assumption, we have $xx = e$, $yy = e$, and $(xy)(xy) = e$.
Then

$$xyxy = (xy)(xy) = e = ee = (xx)(yy) = xxyy$$

Multiply on the left by x and on the right by y:

$$\begin{aligned} x(xyxy)y &= x(xxyy)y \\ (xx)(yx)(yy) &= (xx)(xy)(yy) \\ e(yx)e &= e(xy)e \\ yx &= xy \end{aligned}$$