1 Isomorphism

Consider the map $\varphi : \mathbb{C} \to \mathbb{C}$ defined by

$$\varphi(a + bi) = a - bi$$

(i) Prove that φ preserves addition.

(ii) Prove that φ is surjective.

2 Evaluation homomorphism

Let $\mathbb{Z}[x]$ denote the ring of all polynomials having integer coefficients.
Consider the evaluation homomorphism $\mathrm{ev} : \mathbb{Z}[x] \to \mathbb{Z}$ defined by

$$p(x) \mapsto p(2)$$

(a) What is the kernel of ev?

(b) What is the image of ev?

3 Kernel and 1-to-1 homomorphism

Let $\varphi : R \to S$ be a ring homomorphism. Prove the following: If $\ker \varphi = \{\mathbf{0}_R\}$, then φ is 1 to 1.

4 Matrices with integer entries

Definition 1. Consider the set

$$\mathrm{Mat}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\}$$

of 2×2 matrices with integer entries. It forms a ring with unity $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ under the usual matrix addition and matrix multiplication. The zero element is the zero matrix $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

Let I be the subset of $\mathrm{Mat}_2(\mathbb{Z})$ consisting of matrices with even entries. (You have shown in previous homework that I is a subring). Now, prove that

$$I \text{ is an ideal of } \mathrm{Mat}_2(\mathbb{Z}).$$

5 An ideal of the ring of integer polynomials?

Let $\mathbb{Z}[x]$ denote the ring of all polynomials having integer coefficients. Consider the subset S of $\mathbb{Z}[x]$ of polynomials $f(x)$ such that $f(5) = 0$.

(a) What do the polynomials in S look like? Give some examples.

(b) Is S an ideal?

(c) If S is a principal ideal, describe an element of S which generates S.

6 An ideal of the ring of integer polynomials?

Let $\mathbb{Z}[x]$ denote the ring of all polynomials having integer coefficients. Consider the subset T of $\mathbb{Z}[x]$ of polynomials $f(x)$ such that $f(0) = 5$.

(a) What do the polynomials in S look like? Give some examples.

(b) Is T an ideal?

(c) If T is a principal ideal, describe an element of T which generates T.

7 Ideals?

Let $\mathbb{Z}[x]$ denote the ring of all polynomials having integer coefficients. Which of the following subsets of $\mathbb{Z}[x]$ are ideals? Answer **Yes** or **No**.

- If you answer No, provide a specific example of how the subset fails the absorbing property of an ideal or how the subset fails to be an additive subgroup of $\mathbb{Z}[x]$.
- If you answer Yes, explain why the absorbing property holds (you don't need to prove that the subset is an additive group).

(a) $S = \mathbb{Z}$, that is, all the constant polynomials in $\mathbb{Z}[x]$.

(b) The set S of integer polynomials $f(x)$ such that $f(r) \geq 0$ for all real number r (when you graph the polynomial, the curve is always on or above the x-axis).

(c) The set S of integer polynomials $f(x)$ such that $f(1) \neq 0$, i.e. 1 is not a root of $f(x)$.

(d) The set S of integer polynomials $f(x)$ such that $f'(2) = 0$, i.e. 2 is a root of $f(x)$.

(e) The set S of integer polynomials $f(x)$ whose coefficients are all even integers.

8 An ideal of the ring of real polynomials

Consider the ring $\mathbb{R}[x]$ of polynomials with real coefficients, and let I denote the set of polynomials in $\mathbb{R}[x]$ with no constant term or term of degree 1. For example,

$$p(x) = \pi x^2 - e x^5 \in I,$$

but

$$q(x) = \pi - e x^5 \text{ and } r(x) = \pi x + x^2 \text{ are not in } I.$$

Then I is an ideal of $\mathbb{R}[x]$. Is I a principal ideal? If it is, describe an element of I which generates I.