Section 16.4 Part I: Maximal ideals

1 Question

Demonstrate that the ideal $10\mathbb{Z}$ is not a maximal ideal of $\mathbb{Z}$ by providing another ideal J of $\mathbb{Z}$ which properly contains $10\mathbb{Z}$.

Solution: The ideal $5\mathbb{Z}$ properly contains $10\mathbb{Z}$.

2 Question

The quotient ring $\mathbb{R}[x]/\langle x-3\rangle$ is isomorphic to $\mathbb{R}$ by applying the 1st isomorphism theorem using the evaluation homomorphism

$$\begin{aligned} \varphi : \mathbb{R}[x] &\to \mathbb{R} \quad \text{defined by} \\ p(x) &\mapsto p(3). \end{aligned}$$

Solution: The kernel of φ is the principal ideal $\langle x-3\rangle$ generated by $x-3$. The image of φ is $\mathbb{R}$, since for any $r \in \mathbb{R}$, the constant polynomial $f(x) = r$ is sent to r by φ.
By the 1st isomorphism theorem, $\mathbb{R}[x]/\langle x-3\rangle$ is isomorphic to $\mathbb{R}$.

Apply Theorem 16.35 to this situation. What can you say about the principal ideal $\langle x-3\rangle$? Is it a maximal ideal in $\mathbb{R}[x]$?

Solution: Since $\mathbb{R}$ is a field, the quotient ring $\mathbb{R}[x]/\langle x-3\rangle$ is a field. Because of Theorem 16.35, the ideal $\langle x-3\rangle$ is maximal in $\mathbb{R}[x]$.

3 Question

We demonstrated that the ideal $\langle x\rangle$ of $\mathbb{Z}[x]$ is *not* maximal by providing another ideal $J = \{f(x) \in \mathbb{Z}[x] : f(0) \text{ is an even integer}\}$ such that $\langle x\rangle \subsetneq J$. For example, $x+8 \in J$ but $x+8 \neq \langle x\rangle$.

Let's now give an alternative explanation for why $\langle x\rangle$ is not a maximal ideal of $\mathbb{Z}[x]$.

The quotient ring $\mathbb{Z}[x]/\langle x\rangle$ is isomorphic to $\mathbb{Z}$ by applying the 1st isomorphism theorem using the evaluation homomorphism

$$\begin{aligned} \varphi : \mathbb{Z}[x] &\to \mathbb{Z} \quad \text{defined by} \\ p(x) &\mapsto p(0). \end{aligned}$$

Solution: The kernel of φ is the principal ideal $\langle x \rangle$ generated by x. The image of φ is $\mathbb{Z}$, since for any $z \in \mathbb{Z}$, the constant polynomial $f(x) = z$ is sent to z by φ.
By the 1st isomorphism theorem, $\mathbb{R}[x]/\langle x - 3 \rangle$ is isomorphic to $\mathbb{R}$.

Apply Theorem 16.35 to this situation. What can you say about the principal ideal $\langle x \rangle$? Is it a maximal ideal in $\mathbb{Z}[x]$?

Solution: Since $\mathbb{Z}$ is not a field, the quotient ring $\mathbb{R}[x]/\langle x - 3 \rangle$ is not a field. Because of Theorem 16.35, the ideal $\langle x - 3 \rangle$ is *not* maximal in $\mathbb{Z}[x]$.

4 Question

Note: Question 3 tells us that the principal ideal $I = \langle\ x\ \rangle = \{f(x)x : f(x) \in \mathbb{Z}[x]\}$ is *not* a maximal ideal of $\mathbb{Z}[x]$. However, the principal ideal $J = \langle\ x\ \rangle = \{f(x)x : f(x) \in \mathbb{R}[x]\}$ *is* a maximal ideal of $\mathbb{R}[x]$.

Prove that J is a maximal ideal of $\mathbb{R}[x]$.

Solution: Consider the evaluation homomorphism

$$\begin{aligned} \varphi : \mathbb{R}[x] &\to \mathbb{R} \quad \text{defined by} \\ p(x) &\mapsto p(0). \end{aligned}$$

The kernel of φ is the principal ideal $\langle x \rangle$ generated by x. The image of φ is $\mathbb{R}$, since for any $r \in \mathbb{R}$, the constant polynomial $f(x) = r$ is sent to r by φ. So, by the 1st isomorphism theorem, $\mathbb{R}[x]/\langle x \rangle$ is isomorphic to $\mathbb{R}$.

Since $\mathbb{R}$ is a field, the quotient ring $\mathbb{R}[x]/\langle x \rangle$ is a field. Because of Theorem 16.35, the ideal $\langle x \rangle$ is maximal in $\mathbb{R}[x]$.

Section 16.4 Part II: Prime ideals

5 Question

Prove that $\langle x - 4 \rangle$ is a prime ideal of $\mathbb{Z}[x]$.

Solution: Copy the example in week 14 notes (showing directly that $\langle x \rangle$ is a prime ideal of $\mathbb{Z}[x]$). We now have $\langle\ x - 4\ \rangle = \{g(x) \in \mathbb{Z}[x] : g(4) = 0\}$.

6 Question

Let R be a commutative ring with unity 1. Prove that if I is a maximal ideal of R, then I is also a prime ideal of R.

7 Question

(a) Write the definition of a prime ideal.

(b) Demonstrate that $6\mathbb{Z}$ is not a prime ideal of $\mathbb{Z}$. (Hint: find a pair of elements a, b such that $ab \in 6\mathbb{Z}$ but $a \notin 6\mathbb{Z}$ and $b \notin 6\mathbb{Z}$.)

Solution: $a = 40$, $b = 21$ work.
$a = 22$, $b = 3$ also work.
The simplest choices would be $a = 2$, $b = 3$.